Exercise No 1

ALGORITHMS OF NUMERICAL INTEGRATION OF DIFFERENTIAL EQUATIONS

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1 The aim of experiments

Learning the basic properties of the fixed-step (ode1, ode2, ode3, ode4, ode5) and variable-step numerical integration algorithms (ode45, ode23, ode113, ode15s, ode23s, ode23t, ode23tb). Checking the influence of the choice of the algorithm and the time step for the stability and acuracy of solution, especially at stiff differential equation.

2 Differential equations under test

2.1 Comparison of the algorithms at stiff equation

We take the stiff equation in the form

$$\frac{dx}{dt} = \lambda_2 e^{-\lambda_2 t} + \lambda_1 \left(1 - e^{-\lambda_2 t} - x\right) \tag{1}$$

The coefficients λ_1 and λ_2 represent the time constants. Their relation $k = \frac{\lambda_1}{\lambda_2}$ represents the stiffness of the equation. The higher is the value of k the more difficult problem to solve. Fig. 1 presents the Simulink model of this equation

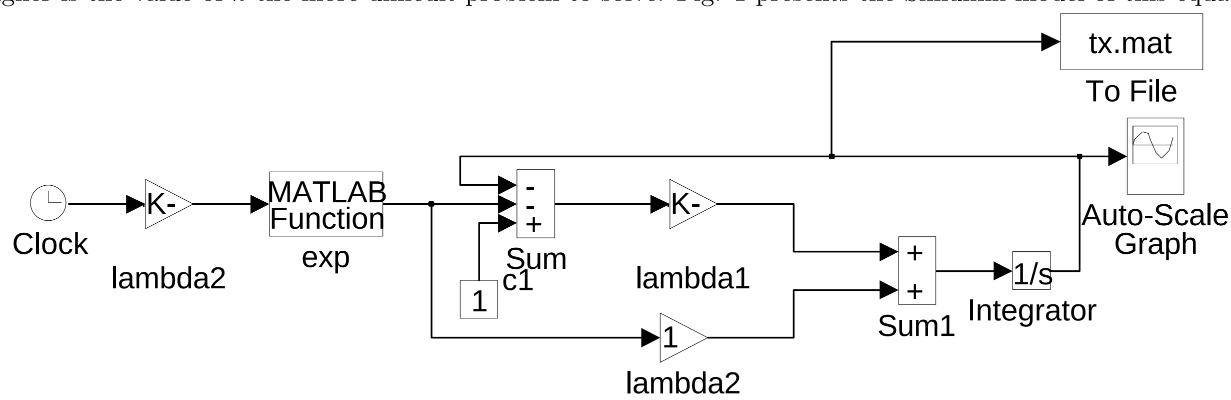

Figure 1: *Model of the stiff equation*

The solution of the equation in the form of vector **tx**= [t, x] is stored additionally in the mat-file *tx*. On the other hand this particular differential equation has the solution of explicitly known form

$$x(t) = x(0)e^{-\lambda_1 t} + \left(1 - e^{-\lambda_2 t}\right) \tag{2}$$

2.2 Comparison of algorithms for nonlinear van der Pol equation

Van der Pol equation is well known nonlinear differential equation used in description of electronic generator built on the basis of the tunnel diode.

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} x_2 \\ \varepsilon(1 - x_1^2)x_2 - x_1 \end{bmatrix}$$

The variable x_1 represents the voltage over the tunnel diode. The Simulink model of this equation is presented in Fig. 2. Changing the value of the coefficient ε we can generate different forms of output signal of generator. At ε close to 0 the signal is sinusoidal. Increasing the value of ε results in the signals approaching the square wave.

3 Program of numerical experiments

3.1 Stiff equation

Applying different values of stiffness coefficient k compare different variable-step algorithms at the initial condition $x(0) = 1$. Use for example

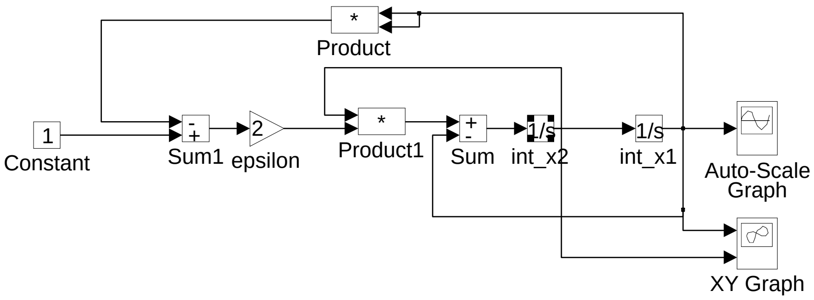

Figure 2: *Simulink model of the van der Pol equation*

- $\lambda_1 = 10, \lambda_2 = 1$
- $\lambda_1 = 10^4, \lambda_2 = 1$
- $\lambda_1 = 10^{10}, \lambda_2 = 1$

The typical form of solution of the equation at the assumed value of initial condition is presented in Fig. 3. In

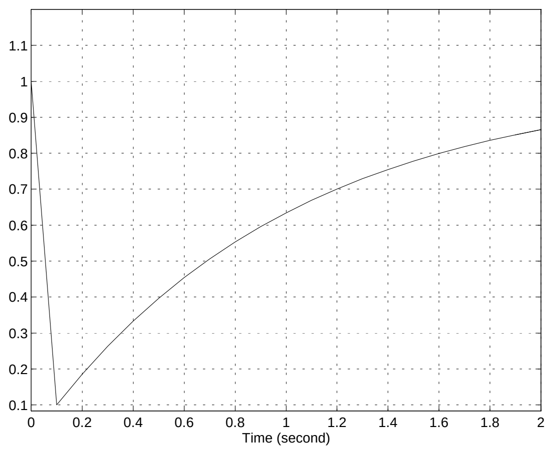

Figure 3: *Exemplary solution of stiff equation at* $k = 10$

particular do the following steps:

1. Build the Simulink model of the stiff equation.
2. Try to solve the equation using fixed-step and variable-step algorithms at three different stiffness parameter k.
3. At $k = 10$ and $k = 10^4$ estimate the time needed to solve the equation using different variable-step algorithms.
4. Setting $k = 100$ compare the stability of fixed-step algorithms at different step length.
5. For one (stable) case of fixed-step algorithm calculate the exact solution and compare it with the results written in the mat-file. Plot the differences of both solution as a function of time

3.2 Van der Pol equation

In this case use the file **vdpol.mdl** in the path $z:\backslash kma\backslash spd)$ implementing the Simulink model of the equation.

1. At different values of ε, for example ε equal 0.1, 0.8, 5, 100) observe the time solution of the equation. Use *ode45* method of integration. Compare the solution in the phase space (the trajectory $dx_1/dt = f(x_1)$)
2. For $\varepsilon = 100$ compare the stability and time needed for the solution of the equation of different algorithms of integration at variable-step length. Try to solve the equation using fixed-step solver.