Electromagnetic Fields

Lecture 11

Waves

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Maxwell equations

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

Moving charges “produce” magnetic field

$$\nabla \times \boldsymbol{E} = \frac{-\partial \boldsymbol{B}}{\partial t}$$

Time-variation of magnetic field causes “produces” curled electric field

$$\nabla \cdot \boldsymbol{D} = \rho$$
$$\nabla \cdot \boldsymbol{B} = 0$$

Electric field originates from charges
There are no magnetic charges

$$\boldsymbol{D} = \varepsilon \boldsymbol{E}$$
$$\boldsymbol{J} = \sigma \boldsymbol{E}$$
$$\boldsymbol{B} = \mu \boldsymbol{H}$$

Approximate response of materials to the electromagnetic field

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Wave equation for *E*

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

$$\boldsymbol{D} = \varepsilon \boldsymbol{E}$$

$$\boldsymbol{J} = \sigma \boldsymbol{E}$$

Substitute ***E*** for ***J*** and ***D***

$$\nabla \times \boldsymbol{H} = \boldsymbol{\sigma} E + \varepsilon \frac{\partial \boldsymbol{E}}{\partial t}$$

Rotate

$$\nabla \times \boldsymbol{E} = \frac{-\partial \boldsymbol{B}}{\partial t}$$

Electric field originates from charges
There are no magnetic charges

$$\boldsymbol{D} = \varepsilon \boldsymbol{E}$$

$$\boldsymbol{J} = \sigma \boldsymbol{E}$$

$$\boldsymbol{B} = \mu \boldsymbol{H}$$

$$\nabla \cdot \boldsymbol{D} = \rho$$

$$\nabla \cdot \boldsymbol{B} = 0$$

Approximate response of materials
to the electromagnetic field

Wave equation for *E*

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$

$$\boldsymbol{D} = \varepsilon \boldsymbol{E}$$

$$\boldsymbol{J} = \sigma \boldsymbol{E}$$

Substitute ***E*** for ***J*** and ***D***

$$\nabla \times \boldsymbol{H} = \sigma \boldsymbol{E} + \varepsilon \frac{\partial \boldsymbol{E}}{\partial t} \quad \star$$

Substitute ***H*** for ***B*** and rotate

$$\nabla \times \boldsymbol{E} = \frac{-\partial \boldsymbol{B}}{\partial t}$$

$$\nabla \times \nabla \times \boldsymbol{E} = -\mu \frac{\partial \nabla \times H}{\partial t}$$

Substitute $\star$ for $\nabla \times \boldsymbol{H}$

$$\nabla \times \nabla \times \boldsymbol{E} = -\mu \sigma \frac{\partial \boldsymbol{E}}{\partial t} - \mu \varepsilon \frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$

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Wave equation for *H*

$$\nabla \times \mathbf{E} = \frac{-\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \quad \star$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\mathbf{D} = \varepsilon \mathbf{E}$$

$$\mathbf{J} = \sigma \mathbf{E}$$

$$\nabla \times \nabla \times \mathbf{H} = \sigma \nabla \times \mathbf{E} + \varepsilon \frac{\partial \nabla \times \mathbf{E}}{\partial t}$$

Substitute ***H*** for ***B***

Substitute ***E*** for ***J*** and ***D*** and rotate

Substitute $\star$ for $\nabla \times \mathbf{E}$

$$\nabla \times \nabla \times \mathbf{H} = -\mu \sigma \frac{\partial \mathbf{H}}{\partial t} - \mu \varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2}$$

Wave equation – some math

For any vector function $\boldsymbol{A}$ holds:

$$\nabla \times \nabla \times \boldsymbol{A} = \nabla (\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$$

$\boldsymbol{H}$ is divergence-free:

$$\nabla \cdot \boldsymbol{H} = 0$$

$$\nabla \times \nabla \times \boldsymbol{H} = -\mu \sigma \frac{\partial \boldsymbol{H}}{\partial t} - \mu \varepsilon \frac{\partial^2 \boldsymbol{H}}{\partial t^2}$$

$$\nabla^2 \boldsymbol{H} = \mu \sigma \frac{\partial \boldsymbol{H}}{\partial t} + \mu \varepsilon \frac{\partial^2 \boldsymbol{H}}{\partial t^2}$$

In Cartesian coordinates

$$\nabla^2 H_x = \mu \sigma \frac{\partial H_x}{\partial t} + \mu \varepsilon \frac{\partial^2 H_x}{\partial t^2}$$

$$\nabla^2 H_y = \mu \sigma \frac{\partial H_y}{\partial t} + \mu \varepsilon \frac{\partial^2 H_y}{\partial t^2}$$

$$\nabla^2 H_z = \mu \sigma \frac{\partial H_z}{\partial t} + \mu \varepsilon \frac{\partial^2 H_z}{\partial t^2}$$

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Wave equation – some math
(continued)

For any vector function $\boldsymbol{A}$ holds:

$$\nabla\times\nabla\times\boldsymbol{A}=\nabla\left(\nabla\cdot\boldsymbol{A}\right)-\nabla^2\boldsymbol{A}$$

$\boldsymbol{E}$ is in general notdivergence-free:

$$\nabla\cdot\boldsymbol{E}=\frac{\rho}{\varepsilon}$$

$$\nabla\times\nabla\times\boldsymbol{E}=-\mu\sigma\frac{\partial\boldsymbol{E}}{\partial t}-\mu\varepsilon\frac{\partial^2\boldsymbol{E}}{\partial t^2}$$

$$\nabla^2\boldsymbol{E}=\mu\sigma\frac{\partial\boldsymbol{E}}{\partial t}+\mu\varepsilon\frac{\partial^2\boldsymbol{E}}{\partial t^2}+\nabla\left(\frac{\rho}{\varepsilon}\right)$$

In most cases

$$\rho=0$$

$$\nabla^2\boldsymbol{E}=\mu\sigma\frac{\partial\boldsymbol{E}}{\partial t}+\mu\varepsilon\frac{\partial^2\boldsymbol{E}}{\partial t^2}$$

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Harmonic fields

$$\boldsymbol{E}(\boldsymbol{r},t)=\boldsymbol{E}(\boldsymbol{r})\sin(\omega t+\psi)=Im\left(E(\boldsymbol{r})e^{j(\omega t+\psi)}\right)=Im\left(\underline{\boldsymbol{E}}(\boldsymbol{r})e^{j\omega t}\right)$$

$$\boldsymbol{H}(\boldsymbol{r},t)=\ldots=Im\left(\underline{\boldsymbol{H}}(\boldsymbol{r})e^{j\omega t}\right)$$

$$\underline{\boldsymbol{D}}(\boldsymbol{r}),\ \underline{\boldsymbol{B}}(\boldsymbol{r}),\ \underline{\boldsymbol{J}}(\boldsymbol{r})$$

Maxwell equations for harmonic fields

$$\nabla\times\boldsymbol{H}=\boldsymbol{J}+\frac{\partial \boldsymbol{D}}{\partial t}$$
$$\nabla\times\boldsymbol{E}=\frac{-\partial \boldsymbol{B}}{\partial t}$$
$$\nabla\cdot\boldsymbol{D}=\rho$$
$$\nabla\cdot\boldsymbol{B}=0$$
$$\boldsymbol{D}=\varepsilon\boldsymbol{E}$$
$$\boldsymbol{J}=\sigma\boldsymbol{E}$$
$$\boldsymbol{B}=\mu\boldsymbol{H}$$

→

$$\nabla\times\underline{\boldsymbol{H}}=\underline{\boldsymbol{J}}+j\omega\underline{\boldsymbol{D}}$$
$$\nabla\times\underline{\boldsymbol{E}}=-j\omega\underline{\boldsymbol{B}}$$
$$\nabla\cdot\underline{\boldsymbol{D}}=\underline{\rho}$$
$$\nabla\cdot\underline{\boldsymbol{B}}=0$$
$$\underline{\boldsymbol{D}}=\varepsilon\underline{\boldsymbol{E}}$$
$$\underline{\boldsymbol{J}}=\sigma\underline{\boldsymbol{E}}$$
$$\underline{\boldsymbol{B}}=\mu\underline{\boldsymbol{H}}$$

→

$$\nabla\times\underline{\boldsymbol{H}}=(\sigma+j\omega\varepsilon)\underline{\boldsymbol{E}}$$
$$\nabla\times\underline{\boldsymbol{E}}=-j\omega\mu\underline{\boldsymbol{H}}$$
$$\nabla\times\underline{\boldsymbol{H}}=j\omega\underline{\varepsilon}\,\underline{\boldsymbol{E}}$$
$$\underline{\varepsilon}=\varepsilon-j\frac{\sigma}{\omega}$$

We love symmetry!

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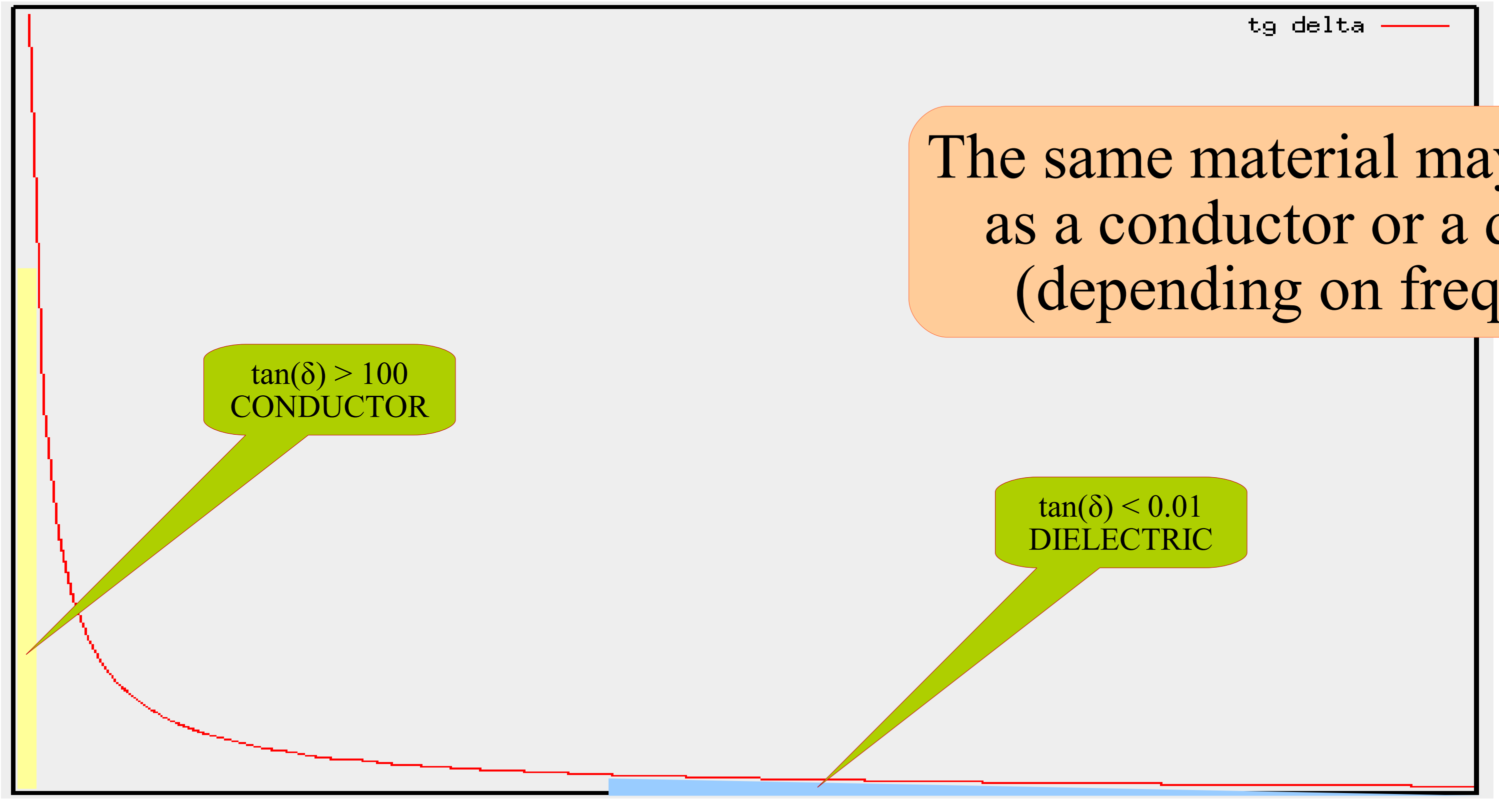
What is “a conductor”?

$$\nabla \times \underline{\boldsymbol{H}} = (\sigma + j\omega\varepsilon)\underline{\boldsymbol{E}} = \underline{\boldsymbol{J}}_\sigma + \underline{\boldsymbol{J}}_\varepsilon \qquad \frac{\underline{\boldsymbol{J}}_\sigma}{\underline{\boldsymbol{J}}_\epsilon} = \frac{\sigma}{\omega\epsilon} = \tan(\delta)$$

$\tan(\delta)$

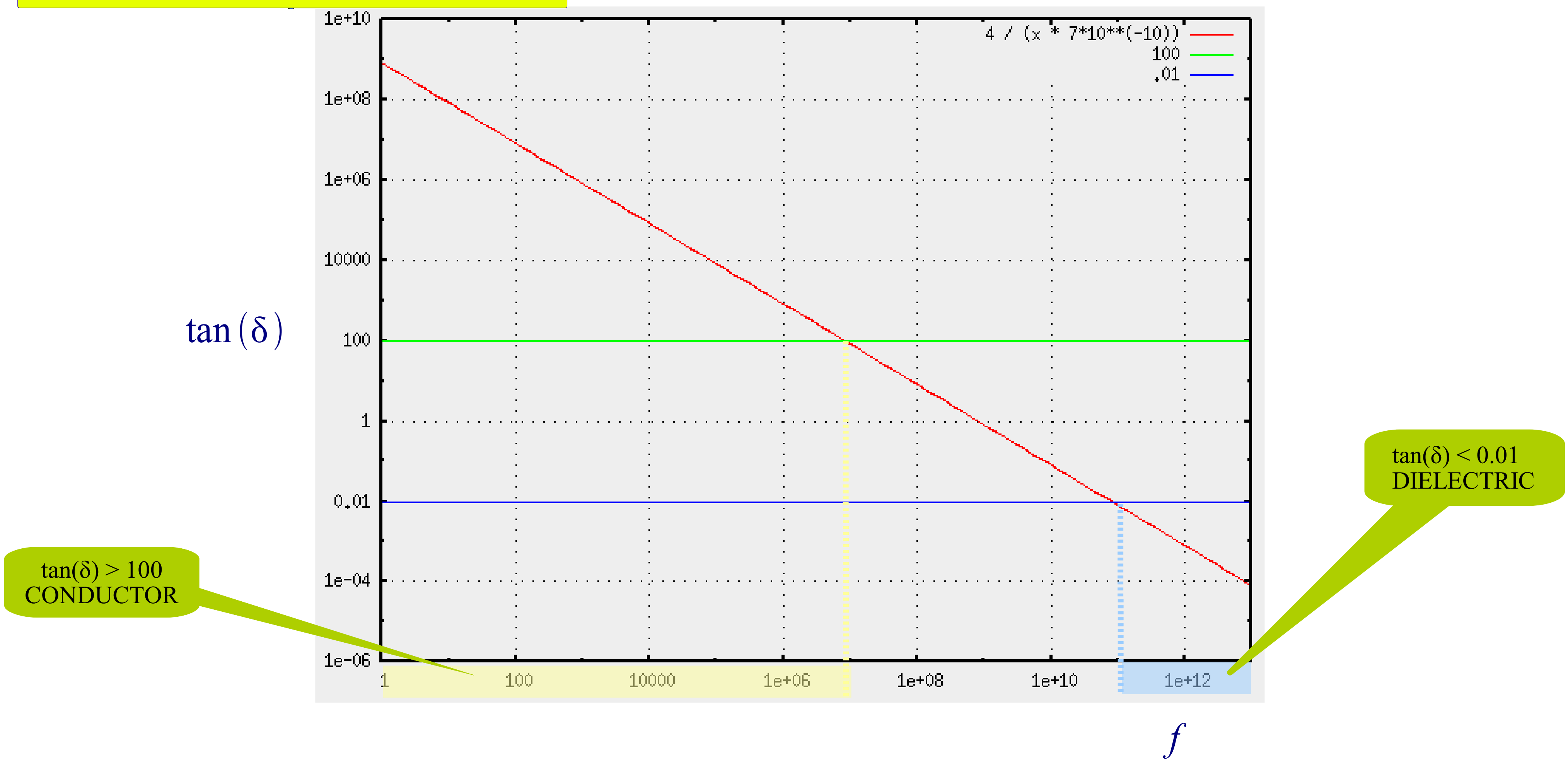


$\omega(f)$

The same material may “behave” as a conductor or a dielectric (depending on frequency).

Example: sea water

$$\sigma = 4\,S/m,\ \varepsilon = 7^{-10}\,F/m$$



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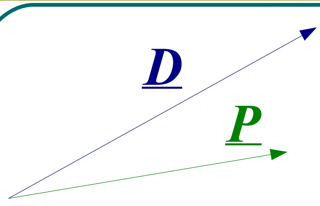
Notes on complex permittivity

$$\underline{\varepsilon} = \varepsilon - j\frac{\sigma}{\omega}$$

$$\underline{\varepsilon} = \varepsilon - j\left(\varepsilon'' + \frac{\sigma}{\omega}\right)$$

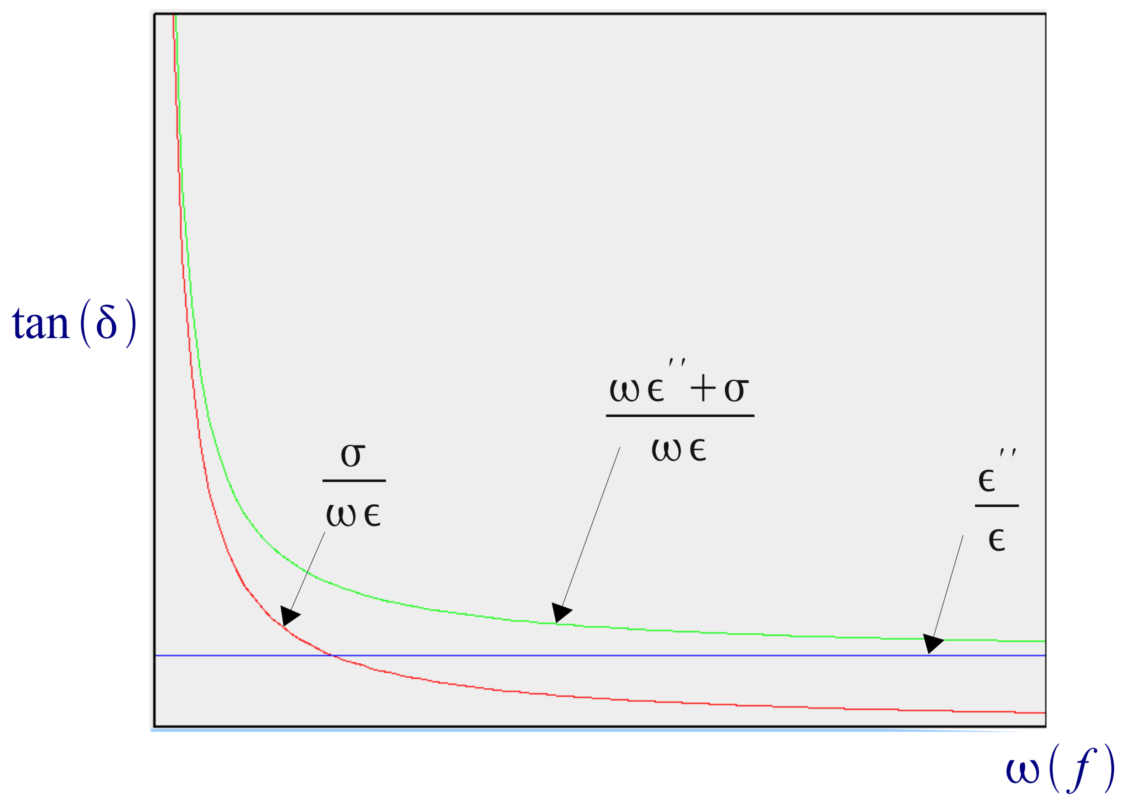
$$\underline{\boldsymbol{J}}_{\sigma} = \left(\sigma + \omega\epsilon''\right)\underline{\boldsymbol{E}}$$

$$\tan(\delta) = \frac{\left(\sigma + \omega\epsilon''\right)}{\omega\epsilon}$$



$$\underline{\boldsymbol{D}} = \left(\epsilon - j\epsilon''\right)\underline{\boldsymbol{E}}$$

For higher frequencies dipoles can not follow the rapid changes of field direction.
We observe a phase shift between the external field and the polarization vector.



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Wave equation – special cases

$$\nabla^2 \boldsymbol{H} = \mu\sigma\frac{\partial \boldsymbol{H}}{\partial t} + \mu\varepsilon\frac{\partial^2 \boldsymbol{H}}{\partial t^2}, \quad \nabla^2 \boldsymbol{E} = \mu\sigma\frac{\partial \boldsymbol{E}}{\partial t} + \mu\varepsilon\frac{\partial^2 \boldsymbol{E}}{\partial t^2} + \nabla\left(\frac{\rho}{\varepsilon}\right)$$

Conductor

1) The conductive current is dominant (due the material properties and relatively slow time-variation of field)
2) No charge accumulation is observed.

$$\nabla^2 \boldsymbol{H} = \mu\sigma\frac{\partial \boldsymbol{H}}{\partial t}$$

$$\nabla^2 \boldsymbol{E} = \mu\sigma\frac{\partial \boldsymbol{E}}{\partial t}$$

These are "diffusion equations", well known form the heat transfer theory

Dielectric

1) The displacement current is dominant (due the material properties or relatively fast time-variation of field)
2) We consider charge free vacuum space.

$$\nabla^2 \boldsymbol{H} = \mu\varepsilon\frac{\partial^2 \boldsymbol{H}}{\partial t^2}$$

$$\nabla^2 \boldsymbol{E} = \mu\varepsilon\frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$

These are "classical" wave equations

$$\frac{1}{\sqrt{\mu_0\varepsilon_0}} = \frac{1}{\sqrt{8.85\times10^{-12}\times4\pi\times10^{-7}}} = 2.99\times10^{8}\,(m/s) = c$$

Harmonic waves

$$\boldsymbol{E}(\boldsymbol{r},t)=\boldsymbol{E}(\boldsymbol{r})\sin(\omega t+\psi)=Im(E(\boldsymbol{r})e^{j(\omega t+\psi)})=Im(\underline{\boldsymbol{E}}(\boldsymbol{r})e^{j\omega t}),\ \underline{\boldsymbol{H}}(\boldsymbol{r}),\ \underline{\boldsymbol{D}}(\boldsymbol{r}),\ \underline{\boldsymbol{B}}(\boldsymbol{r}),\ \underline{\boldsymbol{J}}(\boldsymbol{r})$$

$$\nabla\times\underline{\boldsymbol{H}}=(\sigma+j\omega\varepsilon)\underline{\boldsymbol{E}} \qquad \nabla^2\boldsymbol{H}=j\omega\sigma\mu\underline{\boldsymbol{H}}-\omega^2\epsilon\mu\underline{\boldsymbol{H}}$$

$$\nabla\times\boldsymbol{E}=-j\omega\mu\underline{\boldsymbol{H}} \qquad \nabla^2\boldsymbol{E}=j\omega\sigma\mu\underline{\boldsymbol{E}}-\omega^2\epsilon\mu\underline{\boldsymbol{E}}$$

$$\nabla^2\boldsymbol{H}=\underline{\Gamma}^2\underline{\boldsymbol{H}},\quad \nabla^2\boldsymbol{E}=\underline{\Gamma}^2\underline{\boldsymbol{E}}$$

Helmholtz equations

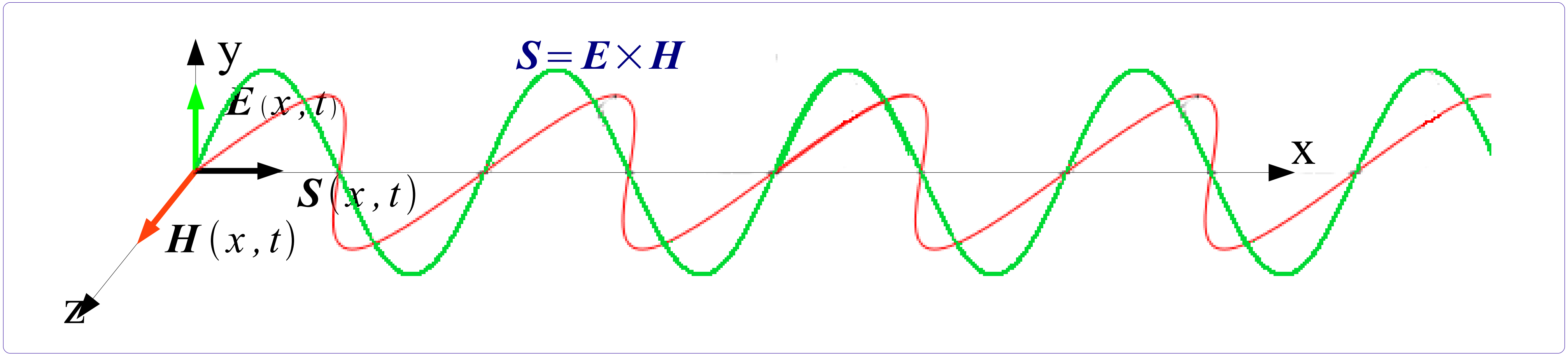
$$\underline{\Gamma}^2=j\omega\sigma\mu-\omega^2\varepsilon\mu=-\omega^2\underline{\varepsilon}\mu \qquad \underline{\epsilon}=\epsilon-j\frac{\sigma}{\omega}$$

$$\underline{\Gamma}=\sqrt{j\omega\sigma\mu-\omega^2\varepsilon\mu}=\alpha+j\beta$$

Solution

$$\nabla^2\boldsymbol{A}=\underline{\Gamma}^2\underline{\boldsymbol{A}} \quad or \quad (\nabla^2+\underline{\Gamma}^2)\boldsymbol{A}=0 \quad \rightarrow \quad \boldsymbol{A}(\boldsymbol{r})=C_1e^{j\underline{\Gamma}\cdot r}+C_2e^{-j\underline{\Gamma}\cdot r}$$

Harmonic plane wave



$$\boldsymbol{H}(\boldsymbol{r},t)=\begin{bmatrix}0 & 0 & H_z(x,t)\end{bmatrix},\qquad \underline{\boldsymbol{H}}=\begin{bmatrix}0 & 0 & \underline{H}_z(x)\end{bmatrix}$$

$$\boldsymbol{E}(\boldsymbol{r},t)=\begin{bmatrix}0 & E_y(x,t) & 0\end{bmatrix},\qquad \underline{\boldsymbol{E}}=\begin{bmatrix}0 & \underline{E}_y(x) & 0\end{bmatrix}$$

$$\frac{\partial^2 \underline{E}_y}{\partial x^2}-\underline{\Gamma}^2\underline{E}_y=0 \qquad \frac{\partial^2 \underline{H}_z}{\partial x^2}-\underline{\Gamma}^2\underline{H}_z=0$$

Incident wave

Reflected wave

$$\underline{E}_y(x)=\underline{E}_1 e^{-\Gamma x}+\underline{E}_2 e^{\Gamma x} \qquad \underline{H}_z(x)=\underline{H}_1 e^{-\Gamma x}+\underline{H}_2 e^{\Gamma x}$$

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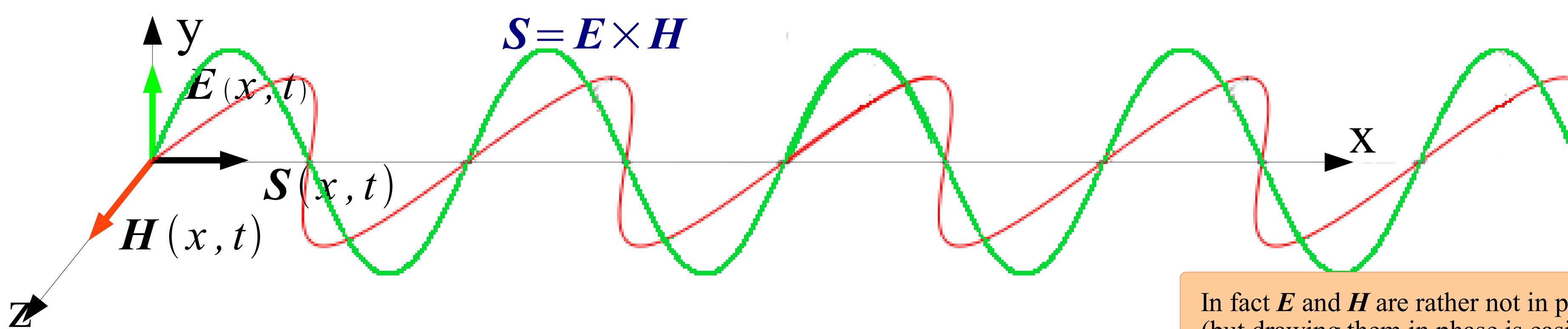
Harmonic plane wave



In fact ***E*** and ***H*** are rather not in phase!
(but drawing them in phase is easier ;-)

$$\underline{\Gamma} = \sqrt{j\omega\sigma\mu - \omega^2 \epsilon\mu} = \alpha + j\beta$$

$$\alpha = \omega\sqrt{\frac{\epsilon\mu}{2}\left(\sqrt{1+\tan\delta} - 1\right)} \qquad \beta = \omega\sqrt{\frac{\epsilon\mu}{2}\left(\sqrt{1+\tan\delta} + 1\right)} \qquad \tan\delta = \frac{\sigma}{\omega\epsilon}$$

$$E_y(x,t) = E_1 e^{-\alpha x}\sin(\omega t - \beta x + \psi_{e1}) + E_2 e^{\alpha x}\sin(\omega t + \beta x + \psi_{e2})$$

$$H_z(x,t) = H_1 e^{-\alpha x}\sin(\omega t - \beta x + \psi_{h1}) + H_2 e^{\alpha x}\sin(\omega t + \beta x + \psi_{h2})$$

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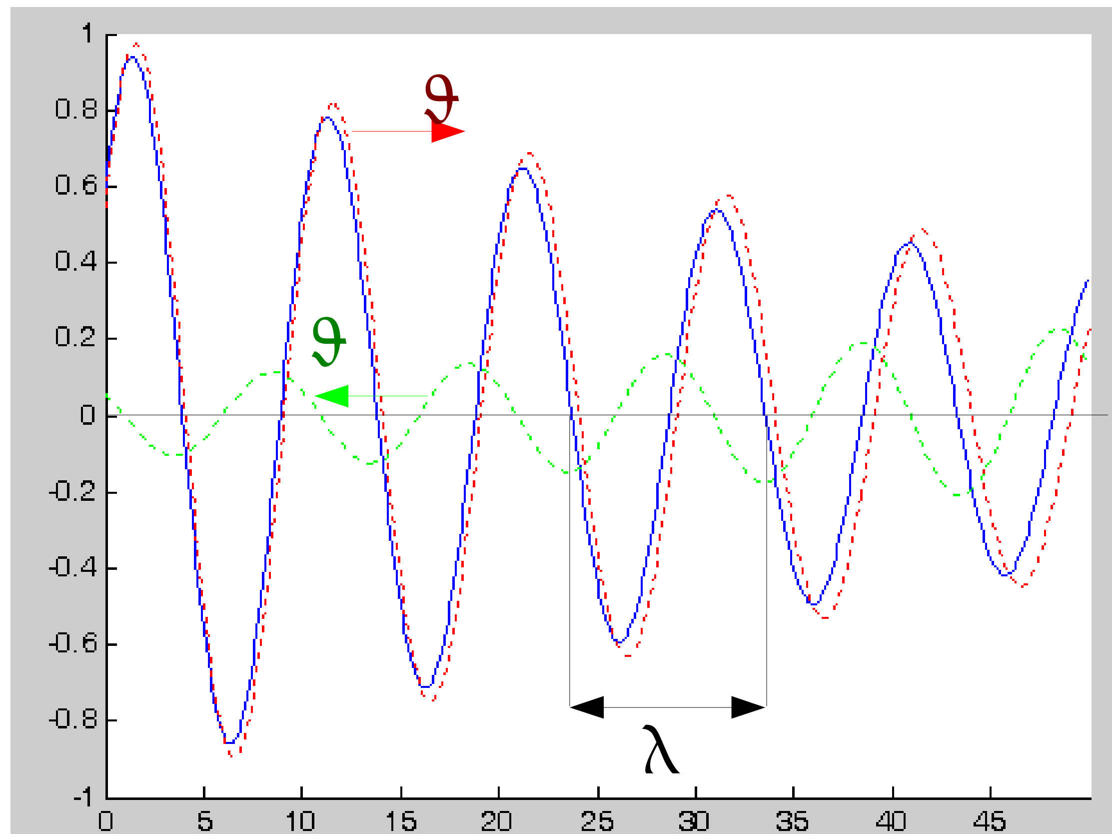
Wave Propagation

$$E_y(x,t)=E_1\, e^{-\alpha x}\sin(\omega t-\beta x+\psi_{e1})+E_2\, e^{\alpha x}\sin(\omega t+\beta x+\psi_{e2})$$

$$\omega t-\beta x+\psi_{e1}=const\,,\qquad \omega t+\beta x+\psi_{e1}=const$$

$$\vartheta=\frac{\partial x}{\partial t}=\frac{\omega}{\beta}$$

$$\lambda=\vartheta\, T=\frac{\vartheta}{f}=\frac{\omega}{\beta}T=\frac{2\pi}{\beta}$$



HPWave in Dielectric

$$\underline{\Gamma} = \sqrt{j\omega\sigma\mu - \omega^2\varepsilon\mu} = \sqrt{-\omega^2\varepsilon\mu} = j\omega\sqrt{\varepsilon\mu}$$

$$E_y(x,t) = E_1 \sin(\omega t - \beta x + \psi_{e1}) + E_2 \sin(\omega t + \beta x + \psi_{e2})$$

$$\beta = \omega\sqrt{\varepsilon\mu}$$

$$Z_c = \frac{E}{H} = \frac{j\omega\mu}{j\omega\sqrt{\varepsilon\mu}} = \sqrt{\frac{\mu}{\varepsilon}}$$

$$\vartheta = \frac{\omega}{\beta} = \frac{1}{\sqrt{\varepsilon\mu}}$$

Vacuum: $\varepsilon_0 = 8.85\,10^{-12}$, $\mu_0 = 4\pi\,10^{-7}$

$Z_c \approx 377\ \Omega$ $\quad \vartheta = c \approx 3\ 10^8\ m/s$

$f = 300\ MHz \rightarrow \lambda \approx 1\ m$

HPWave in Conductor

$$\underline{\Gamma} = \sqrt{j\omega\sigma\mu - \omega^2\varepsilon\mu} \approx \sqrt{j\omega\sigma\mu} = \sqrt{\frac{\omega\sigma\mu}{2}} + j\sqrt{\frac{\omega\sigma\mu}{2}}$$

$$\alpha = \beta = \sqrt{\frac{\omega\sigma\mu}{2}}$$

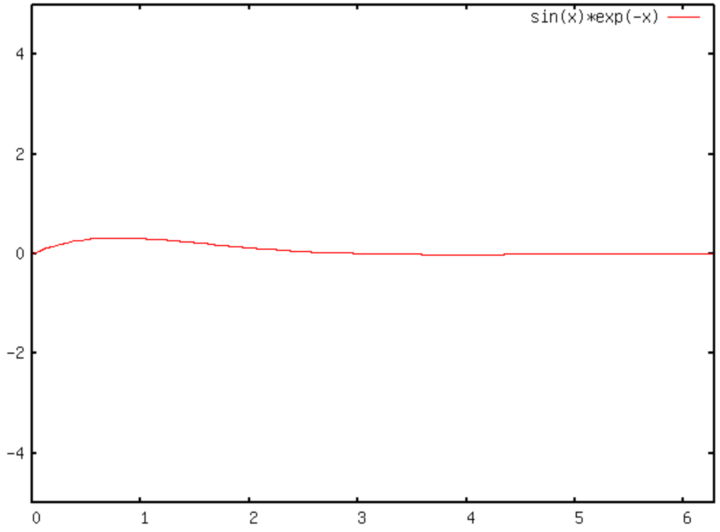
$$\vartheta = \frac{\omega}{\beta} = \sqrt{\frac{2\omega}{\sigma\mu}} \qquad \lambda = \frac{2\pi}{\beta} = 2\sqrt{\frac{\pi}{f\sigma\mu}}$$

$$Z_c = \frac{E}{H} = \frac{j\omega\mu}{\Gamma} = \sqrt{\frac{\omega\mu}{\sigma}}\, e^{j\pi/4}$$

HPWave in Conductor

$$E_y(x,t)=E_1 e^{-\alpha x}\sin(\omega t-\beta x+\psi_{e1})$$

$$e^{-2\pi}\approx 0.0018$$



Field (wave) is rapidly diminishing in a conductor

Skin depth:

$$d=\frac{1}{\alpha}=\sqrt{\frac{2}{\omega\sigma\mu}}=\sqrt{\frac{2}{2\pi f\sigma\mu}}=\frac{\lambda}{2\pi}$$

Skin depth

Conductor	Skin depth at 50 Hz (mm)	Skin depth at 1GHz(μm)
Steel Very approximate!	2.26	0.16
Aluminum	11.74	0.83
Copper	9.48	0.67
Gold	11.17	0.79
Silver	9.05	0.64

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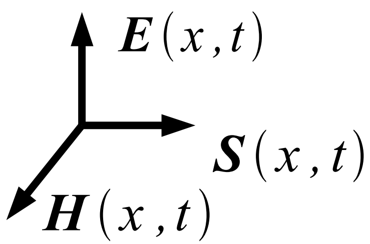
HPW on the Material Boundary



Z_{c1} | Z_{c2}

$$\underline{E}_1 = \underline{E}_{1p} e^{-\Gamma_1 x} + \underline{E}_{1o} e^{\Gamma_1 x}$$

$$\underline{H}_1 = \underline{H}_{1p} e^{-\Gamma_1 x} + \underline{H}_{1o} e^{\Gamma_1 x}$$

$$\underline{E}_1 / \underline{H}_1 = Z_{c1}$$

$$\underline{E}_2 = \underline{E}_{2p} e^{-\Gamma_2 x}$$

$$\underline{H}_2 = \underline{H}_{2p} e^{-\Gamma_2 x}$$

$$\underline{E}_2 / \underline{H}_2 = Z_{c2}$$

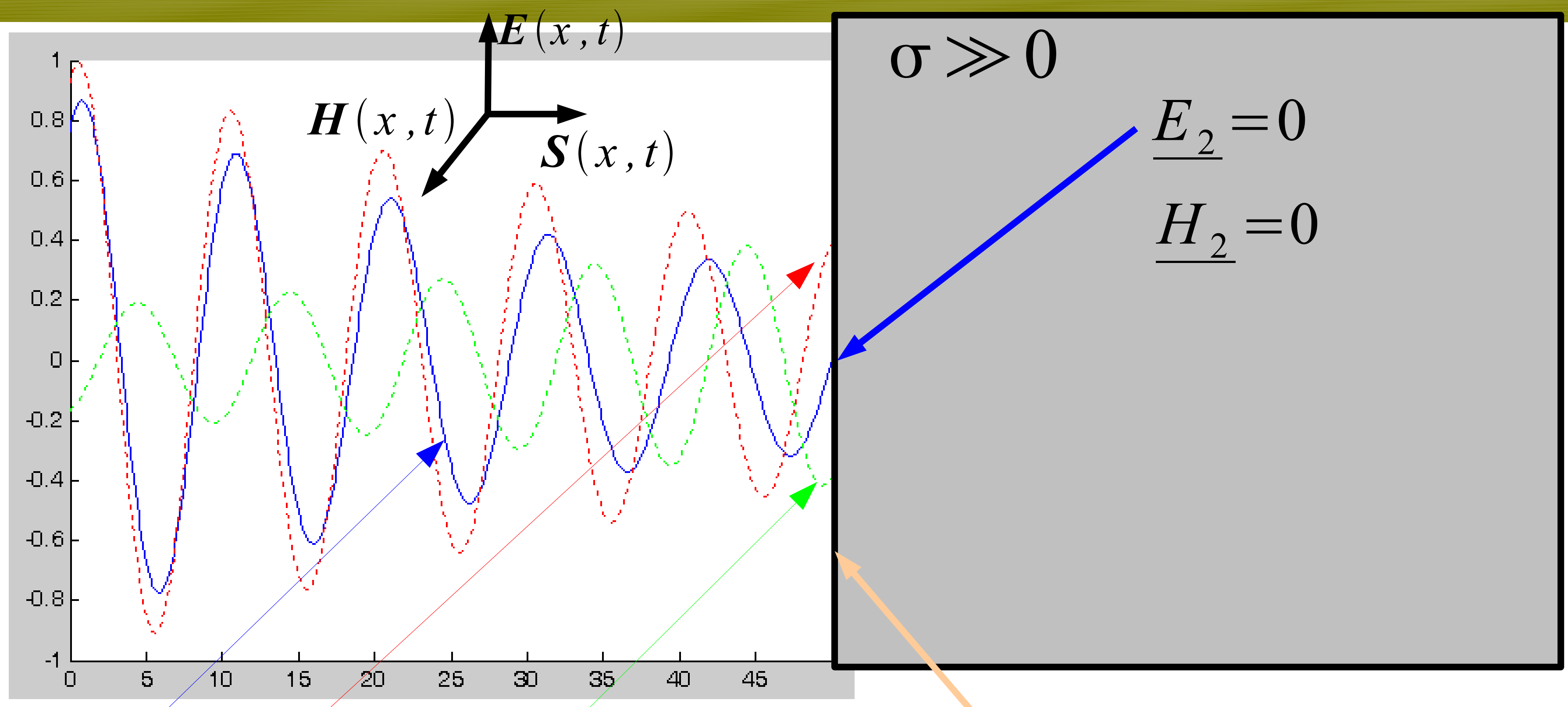
$$\underline{E}_{1p} + \underline{E}_{1o} = \underline{E}_{2p}$$

$$\underline{H}_{1p} + \underline{H}_{1o} = \underline{H}_{2p}$$

$$\underline{E}_{1o} = M\,\underline{E}_{1p} \qquad \underline{H}_{1o} = -M\,\underline{H}_{1p} \qquad M = \frac{Z_{c2} - Z_{c1}}{Z_{c2} + Z_{c1}}$$

$$\underline{E}_{2p} = N\,\underline{E}_{1p} \qquad \underline{H}_{2p} = Z_{c1}/Z_{c2}\,N\,\underline{H}_{1p} \qquad N = \frac{2\,Z_{c2}}{Z_{c2} + Z_{c1}}$$

HPW on the Conductor Boundary



$$\underline{E}_1 = \underline{E}_{1p} e^{-\Gamma_1 x} + \underline{E}_{1o} e^{\Gamma_1 x}$$

$$\underline{H}_1 = \underline{H}_{1p} e^{-\Gamma_1 x} + \underline{H}_{1o} e^{\Gamma_1 x}$$

$$\underline{E}_{1p} + \underline{E}_{1o} = \underline{E}_{2p}$$

$$\underline{H}_{1p} + \underline{H}_{1o} = \underline{H}_{2p}$$

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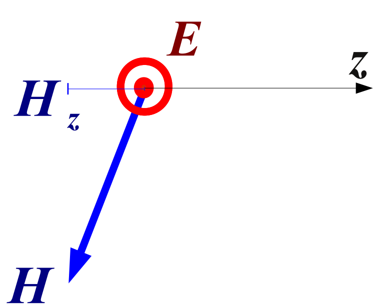
Transverse waves

General Wave propagating in lossless material (α=0) the +z direction:

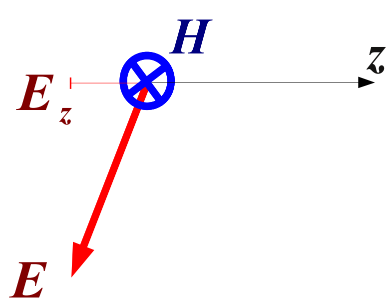
$$\underline{\boldsymbol{E}}=\left(E_x\boldsymbol{x}+E_y\boldsymbol{y}+E_z\boldsymbol{z}\right)e^{-j\beta z}=\left(E_t\boldsymbol{t}+E_z\boldsymbol{z}\right)e^{-j\beta z}$$

$$\underline{\boldsymbol{H}}=\left(H_x\boldsymbol{x}+H_y\boldsymbol{y}+H_z\boldsymbol{z}\right)e^{-j\beta z}=\left(H_t\boldsymbol{t}+H_z\boldsymbol{z}\right)e^{-j\beta z}$$

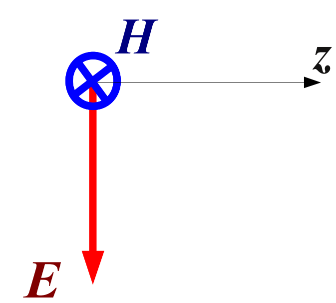
$E_z=0$ → *Transverse Electric*



$H_z=0$ → *Transverse Magnetic*



$E_z=0, H_z=0$ → *Transverse ElectroMagnetic*



Plane wave is a TEM wave

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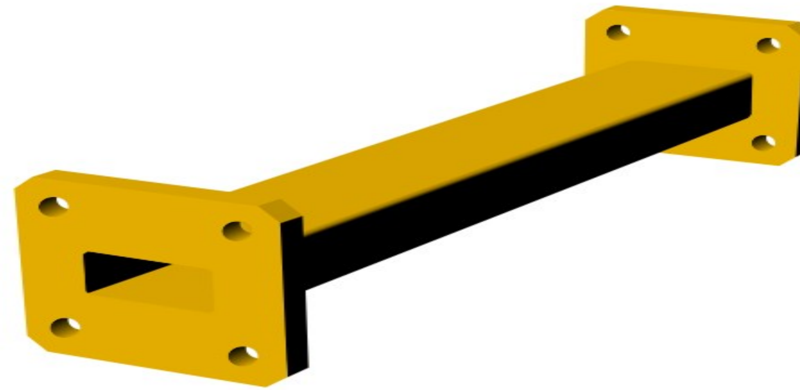
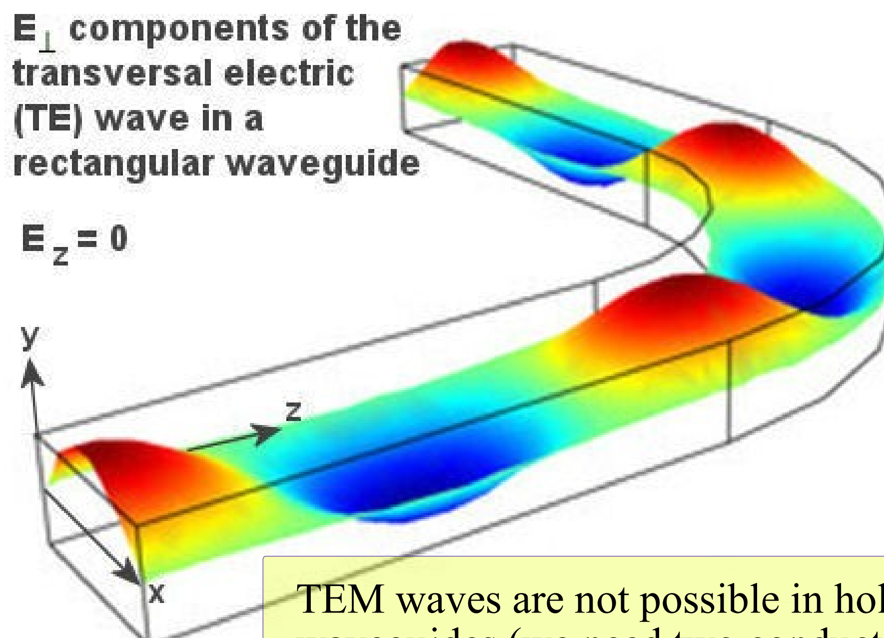
Waveguides

Waveguides are hollow structures (conductive metal pipes of different shapes) which guide waves. Internal walls of nearly all waveguides are plated with high-conductive metals (copper, silver, gold). Close to the conductor the electric field must be perpendicular to it and the magnetic field must be parallel – so the wave is reflected by the waveguide walls and can be imagined as a "zigzag" traveling along the waveguide.

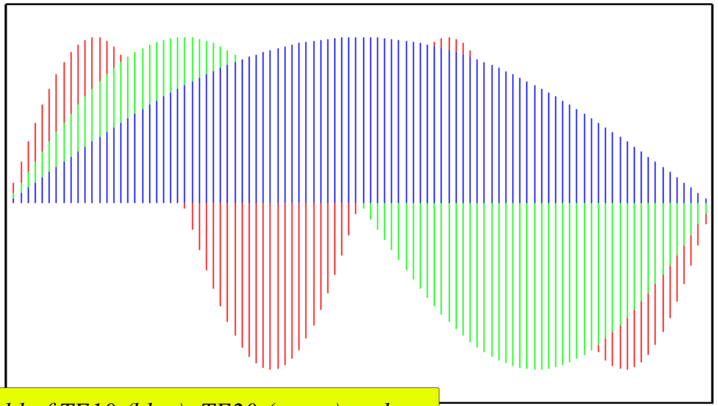




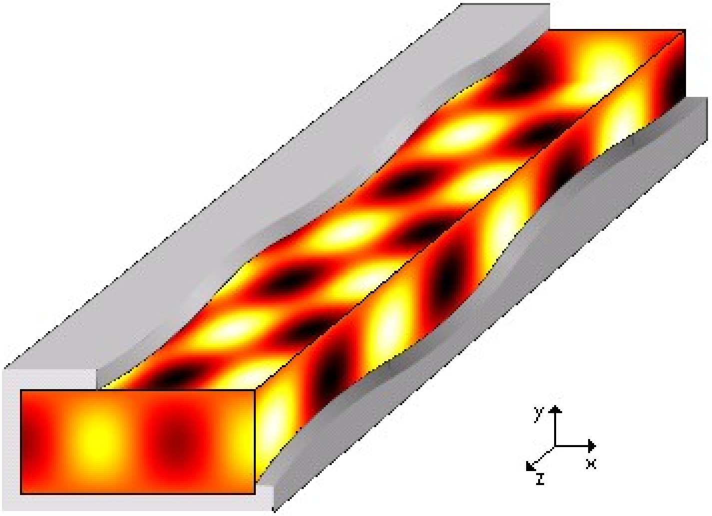
TEM waves are not possible in hollow waveguides (we need two conductors to have TEM wave!).

Waveguide modes

Because of the boundary conditions the Maxwell equations have restricted number of solutions in a waveguide. A *propagation mode* in a waveguide is a particular solution of wave equation. The geometry of a waveguide limits the frequency range in which each of modes can propagate. Each mode has a *cutoff frequency*, below which the propagation is not possible (attenuation is to high).
The mode with the lowest cutoff frequency is called the *dominant mode* of the guide. It is usual to choose the size of the guide such that only this one mode can exist in the frequency band of operation.



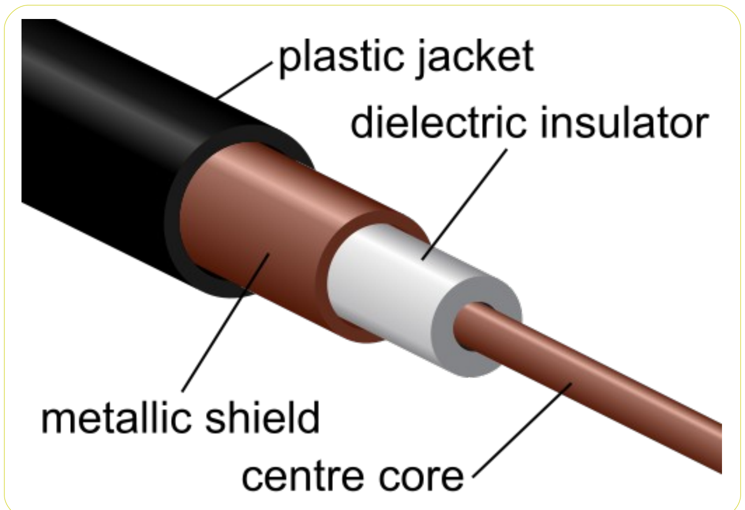
Electric field of TE10 (blue), TE20 (green) and TE30 (red) modes in rectangular waveguide.



*Mode 31 of a rectangular x-band waveguide at 32 GHz.
The electric field is in the x direction.
The waveguide is cut along its side to allow a view of the field inside.
Created using Matlab, OfficeXP, PhotoScape by formulae from
C.A. Balanis:Advanced engineering electromagnetics (Willey, 1989)
Public domain picture from wikipedia*

Coaxial cable

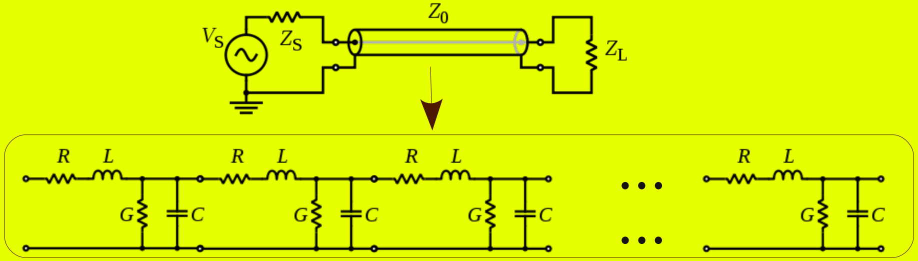
Coaxial cable – an electrical cable with an inner conductor surrounded by a flexible, tubular insulating layer, surrounded by a tubular conducting shield.



In radio-frequency applications up to a few gigahertz, the wave propagates primarily in the TEM mode. Above the cutoff frequency the TE and TM modes are also possible – it is undesirable to operate above the cutoff frequency (roughly speaking the cutoff f. is inversely proportional to the outer diameter of coax).

Characteristic parameters

It is often much more effective to treat waveguides as electric circuits. For this purpose we need unit parameters of the cable: capacitance, inductance and, eventually, resistance.



$$C=\frac{2\pi\varepsilon}{\ln(D/d)}$$

$$R\approx 0,\quad G\approx\infty$$

$$L=\frac{\mu}{2\pi}\ln(D/d)$$

d – outer diameter of the core
D – inner diameter of the shield

$$Z_0=\sqrt{\frac{L}{C}}=\frac{1}{2\pi}\sqrt{\frac{\mu}{\varepsilon}}\ln(D/d)\approx\frac{138}{\sqrt{\varepsilon_r}}\log_{10}(D/d)\Omega$$

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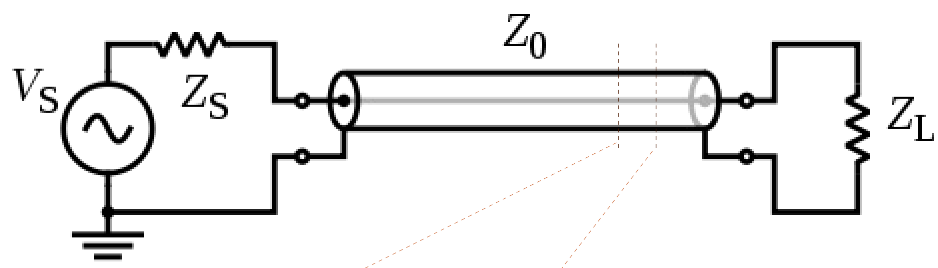
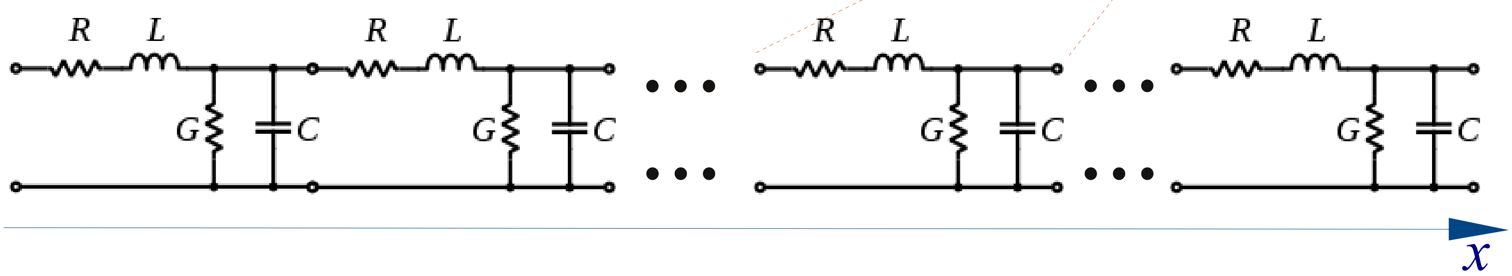
KAPITAŁ LUDZKI NARODOWA STRATEGIA SPÓJNOŚCI
UNIA EUROPEJSKA EUROPEJSKI FUNDUSZ SPOŁECZNY

Transmission line

Some electric circuit can not be treated as real circuits, because their dimension are comparable with the wavelength. In many electric circuits, the length of the wires connecting the components can for the most part be ignored. That is, the voltage on the wire at a given time can be assumed to be the same at all points. However, when the voltage changes in a time interval comparable to the time it takes for the signal to travel down the wire, the length becomes important and the wire must be treated as a *transmission line*.



Telegrapher's equations were published in the modern form by Oliver Heaviside in 1885, but already in 1855 Lord Kelvin formulated a diffusion model of current in submarine cable and predicted poor performance of the 1858 transatlantic submarine telegraph cable.



$$\frac{\partial U(x)}{\partial x} = -(R + j\omega L) I(x)$$

$$\frac{\partial I(x)}{\partial x} = -(G + j\omega C) U(x)$$

$$R \approx 0,\quad G \approx \infty$$

$$\frac{\partial^2 U(x)}{\partial x^2} + \omega^2 LC \cdot U(x) = 0$$

$$\frac{\partial^2 I(x)}{\partial x^2} + \omega^2 LC \cdot I(x) = 0$$

Characteristic parameters

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}, \quad (\gamma = j\omega\sqrt{LC}) \quad \underline{Z}_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}, \quad \left(\underline{Z}_0 = \sqrt{\frac{L}{C}}\right)$$

Solution of telegrapher's eqtn.

$$\frac{\partial^2 \underline{U}(x)}{\partial x^2} = \gamma^2 \cdot \underline{U}(x)$$

$$\frac{\partial^2 \underline{I}(x)}{\partial x^2} = \gamma^2 \cdot \underline{I}(x)$$

$$\underline{U}(x) = \underline{U}^+ e^{-\gamma x} + \underline{U}^- e^{\gamma x}$$

$$\underline{I}(x) = \frac{\underline{U}^+ e^{-\gamma x} - \underline{U}^- e^{\gamma x}}{\underline{Z}_0}$$

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}, \quad (\gamma = j\omega\sqrt{LC})$$

$$\underline{Z}_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}, \quad \left(\underline{Z}_0 = \sqrt{\frac{L}{C}}\right)$$

$$u(x,t) = \sqrt{2}\, U^+ e^{-\alpha x} \sin(\omega t - \beta x + \varphi^+) + \sqrt{2}\, U^- e^{\alpha x} \sin(\omega t + \beta x + \varphi^-)$$

$$i(x,t) = \sqrt{2}\, \frac{V^+}{Z_0} e^{-\alpha x} \sin(\omega t - \beta x + \varphi^+) - \sqrt{2}\, \frac{V^-}{Z_0} e^{\alpha x} \sin(\omega t + \beta x + \varphi^-)$$

Let voltage and current at the line beginning is given as $\underline{U}(x=0) = \underline{U}_1, \quad \underline{I}(x=0) = \underline{I}_1$

$$\underline{U}_1 = \underline{U}^+ + \underline{U}^-$$

$$\underline{Z}_0 \underline{I}_1 = \underline{U}^+ - \underline{U}^-$$

$$\underline{U}^+ = \frac{\underline{U}_1 + \underline{Z}_0 \underline{I}_1}{2}, \quad \underline{U}^- = \frac{\underline{U}_1 - \underline{Z}_0 \underline{I}_1}{2}$$

Solution of telegrapher's eqtn.

(continued)

$$\underline{U}(x)=\underline{U}^{+}e^{-\gamma x}+\underline{U}^{-}e^{\gamma x}$$

$$\underline{I}(x)=\frac{\underline{U}^{+}e^{-\gamma x}-\underline{U}^{-}e^{\gamma x}}{\underline{Z}_0}$$

$$\underline{U}^{+}=\frac{\underline{U}_1+\underline{Z}_0\underline{I}_1}{2},\quad \underline{U}^{-}=\frac{\underline{U}_1-\underline{Z}_0\underline{I}_1}{2}$$

$$\gamma=\sqrt{(R+j\omega L)(G+j\omega C)},\ \left(\gamma=j\omega\sqrt{LC}\right)\qquad \underline{Z}_0=\sqrt{\frac{R+j\omega L}{G+j\omega C}},\ \left(\underline{Z}_0=\sqrt{\frac{L}{C}}\right)$$

$$\underline{U}(x)=\frac{\underline{U}_1+\underline{Z}_0\underline{I}_1}{2}e^{-\gamma x}+\frac{\underline{U}_1-\underline{Z}_0\underline{I}_1}{2}e^{\gamma x}=\underline{U}_1\frac{e^{\gamma x}+e^{-\gamma x}}{2}-\underline{Z}_0\underline{I}_1\frac{e^{\gamma x}-e^{-\gamma x}}{2}$$

$$\underline{I}(x)=\frac{\underline{U}_1+\underline{Z}_0\underline{I}_1}{2\underline{Z}_0}e^{-\gamma x}-\frac{\underline{U}_1-\underline{Z}_0\underline{I}_1}{2\underline{Z}_0}e^{\gamma x}=-\frac{\underline{U}_1}{\underline{Z}_0}\frac{e^{\gamma x}-e^{-\gamma x}}{2}+\underline{I}_1\frac{e^{\gamma x}+e^{-\gamma x}}{2}$$

$$\underline{U}(x)=\underline{U}_1\cosh(\gamma x)-\underline{Z}_0\underline{I}_1\sinh(\gamma x)$$

$$\underline{I}(x)=-\frac{\underline{U}_1}{\underline{Z}_0}\sinh(\gamma x)+\underline{I}_1\cosh(\gamma x)$$

Voltage and current at distance x from the beginning expressed by the voltage and current at the line beginning.

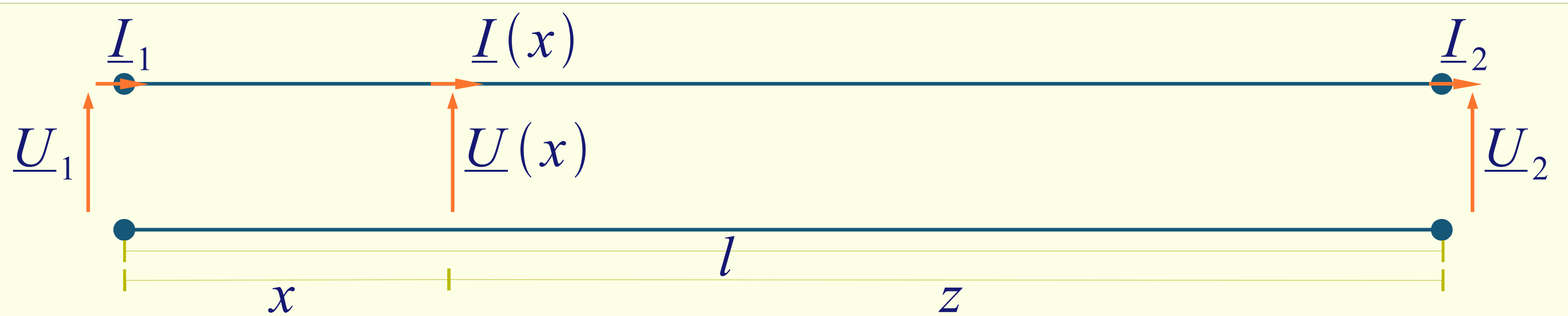
Solution of telegrapher's eqtn.

(continued)

$$\underline{U}(x)=\underline{U}_1\cosh(\gamma x)-\underline{Z}_0\underline{I}_1\sinh(\gamma x)$$

$$\underline{I}(x)=-\frac{\underline{U}_1}{\underline{Z}_0}\sinh(\gamma x)+\underline{I}_1\cosh(\gamma x)$$

Voltage and current at distance x from the beginning expressed by the voltage and current at the line beginning.



$$\underline{U}(z)=\underline{U}_2\cosh(\gamma z)+\underline{Z}_0\underline{I}_2\sinh(\gamma z)$$

$$\underline{I}(z)=\frac{\underline{U}_2}{\underline{Z}_0}\sinh(\gamma z)+\underline{I}_2\cosh(\gamma x)$$

Voltage and current at distance z from the end expressed by the voltage and current at the line end.

Substituting x=l in the firs set or z=l in the second we can get relation between the voltage and current at both ends of the line.

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