Electromagnetic Fields

Lecture 12

Energy

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Energy of electromagnetic field

Electric field

$$W_E = \iiint_V w_E \, dV$$

$$w_E = \frac{\boldsymbol{E} \cdot \boldsymbol{D}}{2} = \frac{\varepsilon E^2}{2} = \frac{D^2}{2\varepsilon}$$

Magnetic field

$$W_H = \iiint_V w_H \, dV$$

$$w_H = \frac{\boldsymbol{H} \cdot \boldsymbol{B}}{2} = \frac{\mu H^2}{2} = \frac{B^2}{2\mu}$$

Total energy

$$W = \iiint_V (w_E + w_H) \, dV =$$

$$= \frac{1}{2} \iiint_V (\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B}) \, dV =$$

$$= \frac{1}{2} \iiint_V (\varepsilon E^2 + \mu B^2) \, dV$$

The electromagnetic energy density

$$w = \frac{\boldsymbol{E} \cdot \boldsymbol{D}}{2} + \frac{\boldsymbol{H} \cdot \boldsymbol{B}}{2}$$

Note that w can only be given if linear, nondispersive and uniform materials are involved, i.e., if

$$\boldsymbol{D} = \varepsilon \boldsymbol{E}$$

$$\boldsymbol{B} = \mu \boldsymbol{H}$$

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Change of energy in time

Consider change of the energy in time

$$\frac{\partial W}{\partial t}=\iiint_V\left(\frac{\partial w_E}{\partial t}+\frac{\partial w_H}{\partial t}\right)dV=\frac{1}{2}\iiint_V\left(\varepsilon\frac{\partial E^2}{\partial t}+\mu\frac{\partial B^2}{\partial t}\right)dV=$$

$$=\iiint_V\left(\varepsilon\,\boldsymbol{E}\frac{\partial \boldsymbol{E}}{\partial t}+\mu\,\boldsymbol{H}\frac{\partial \boldsymbol{H}}{\partial t}\right)dV=$$

Maxwell again!

$$\nabla\times\boldsymbol{H}=\boldsymbol{\sigma}\,E+\varepsilon\frac{\partial \boldsymbol{E}}{\partial t}$$

$$\varepsilon\frac{\partial \boldsymbol{E}}{\partial t}=\nabla\times\boldsymbol{H}-\boldsymbol{\sigma}\,E$$

$$\nabla\times\boldsymbol{E}=-\mu\frac{\partial \boldsymbol{H}}{\partial t}$$

$$\mu\frac{\partial \boldsymbol{H}}{\partial t}=-\nabla\times\boldsymbol{E}$$

$$=\iiint_V\left(\boldsymbol{E}\cdot\nabla\times\boldsymbol{H}-\sigma\,E^2-\boldsymbol{H}\cdot\nabla\times E\right)dV=$$

Vector identity

$$\boldsymbol{E}\,\nabla\times\boldsymbol{H}-\boldsymbol{H}\,\nabla\times\boldsymbol{E}=\nabla\cdot(\boldsymbol{E}\times\boldsymbol{H})$$

$$=-\iiint_V\sigma\,E^2\,dV-\iiint_V\nabla\cdot(\boldsymbol{E}\times\boldsymbol{H})\,dV=$$

Stokes's theorem

$$=-\iiint_V\sigma\,E^2\,dV-\oiint_{\partial V}\boldsymbol{E}\times\boldsymbol{H}\,dS$$

The first term in the right-hand side represents the energy dissipated in form of heat.
The second term represents net electromagnetic energy flow into a small volume.

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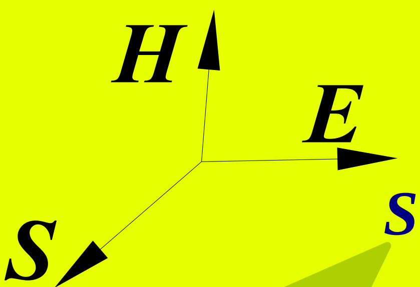
Change of energy in time
(continued)

Interpretation

Power which is dissipated
if form of electric heating.
Change of energy is negative
- energy is “lost” (for EM field;-)

$$\frac{\partial W}{\partial t} = \iiint_V \left(\frac{\partial w_E}{\partial t} + \frac{\partial w_H}{\partial t} \right) dV = - \iiint_V \sigma E^2 \, dV - \oiint_{\partial V} \boldsymbol{E} \times \boldsymbol{H} \, d\boldsymbol{S}$$

Power radiated through the boundary of the small volume V. It is negative or positive depending on direction of $\boldsymbol{E}\times\boldsymbol{H}$ vector with respect to the outward normal vector of ∂V (boundary of V).



$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$$

Poyting vector represents the flux of the energy of an electromagnetic field.

Units: W /m^2

Poynting vector was co-invented independently by John Henry Poynting, Oliver Heaviside and Nikolai Umov. Umov's works were published 10 earlier than Poynting, but he was writing about transfer of energy in elastic media and in the fluid, although we generalized the theory of energy movement.

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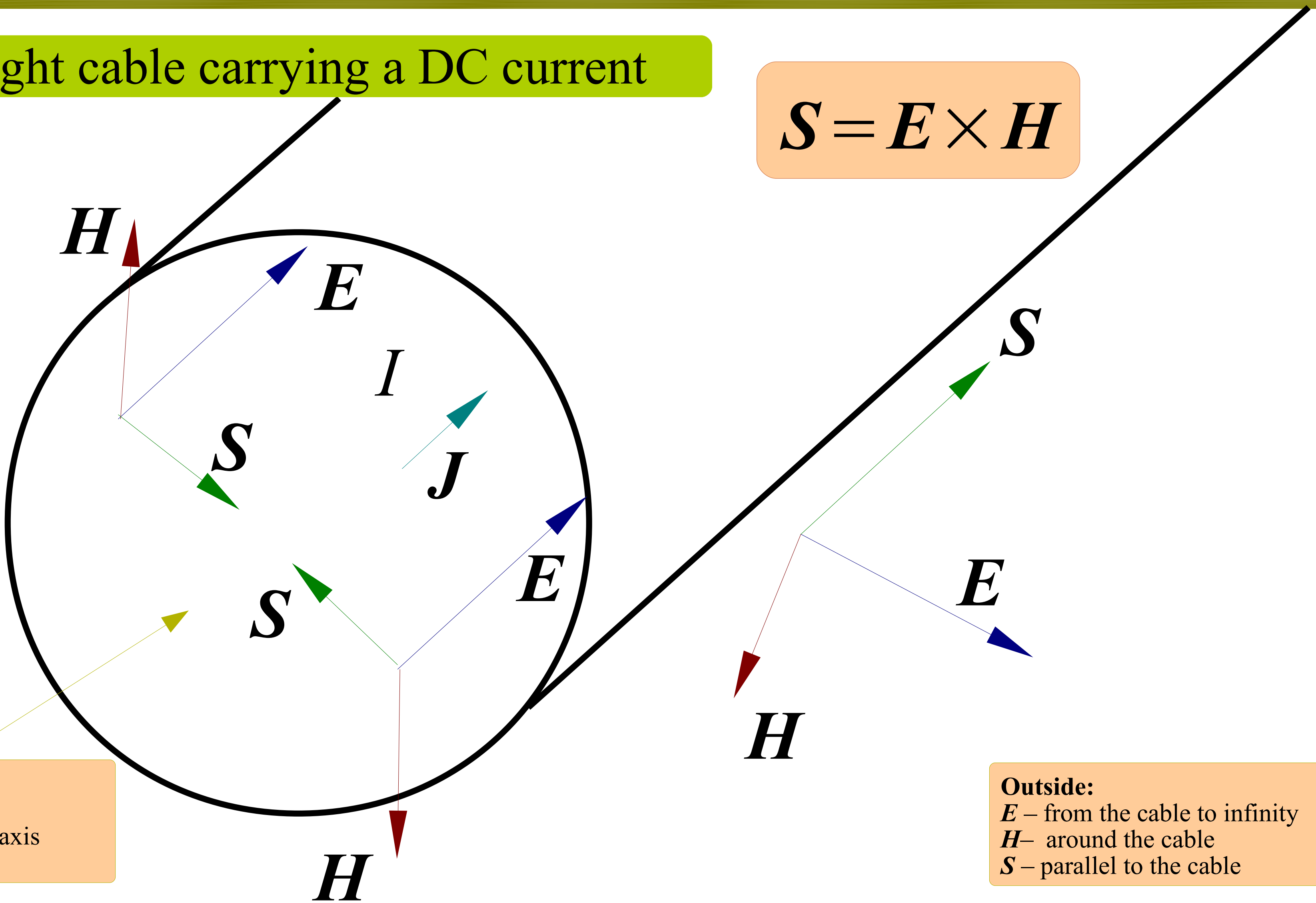
Example

Long, straight cable carrying a DC current

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$$



Inside:
E – along the cable
H– around the central axis
S – into to the cable

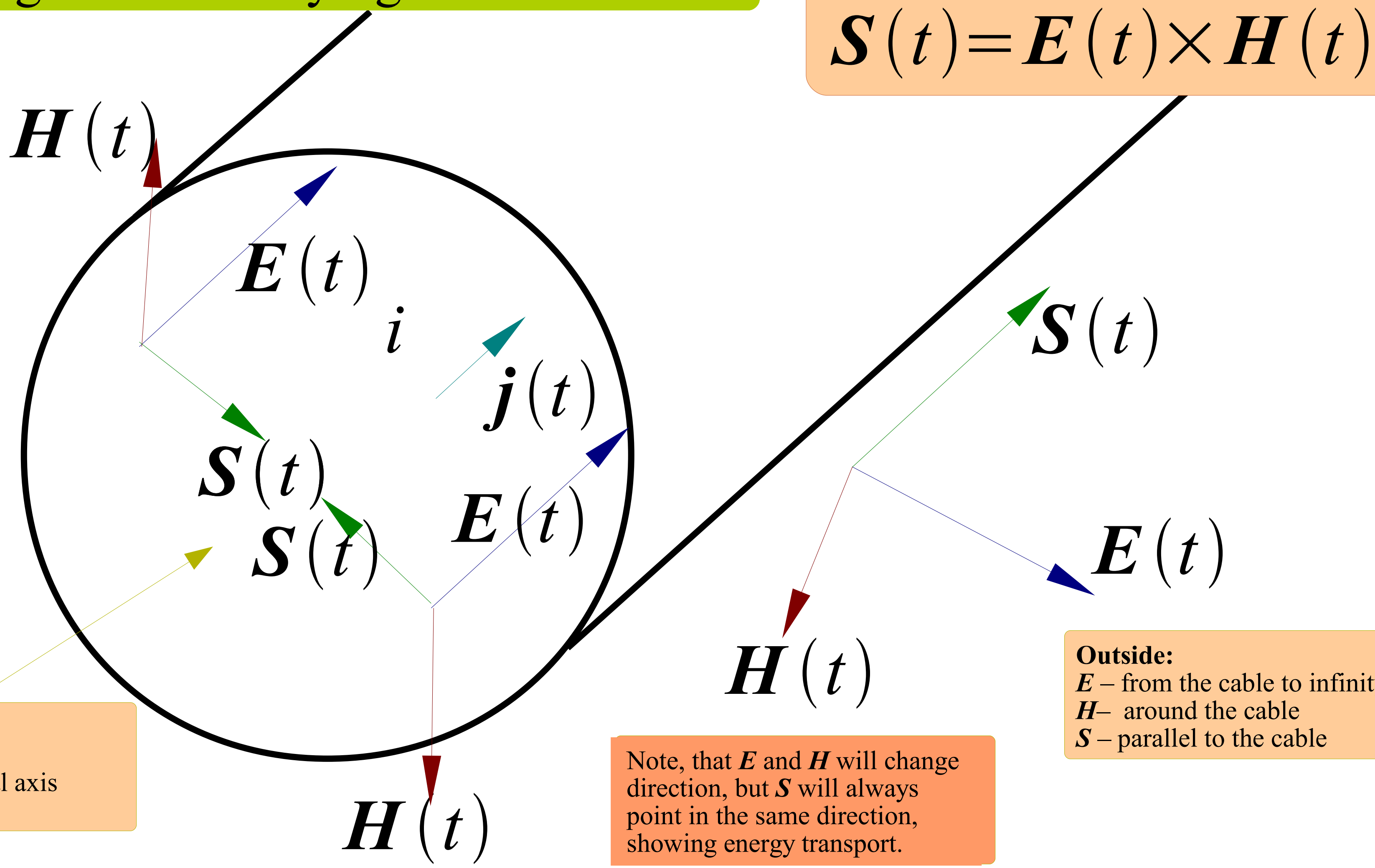
Outside:
E – from the cable to infinity
H– around the cable
S – parallel to the cable

Example

(continued)

Long, straight cable carrying an AC current

$$\boldsymbol{S}(t)=\boldsymbol{E}(t)\times\boldsymbol{H}(t)$$



Inside:
E – along the cable
H– around the central axis
S – into to the cable

Note, that ***E*** and ***H*** will change direction, but ***S*** will always point in the same direction, showing energy transport.

Outside:
E – from the cable to infinity
H– around the cable
S – parallel to the cable

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Example

(continued)

Long, straight cable carrying an DC current *I*

Inside:

$$\boldsymbol{J} = J\,\boldsymbol{z}\,, \quad J = \frac{I}{\pi R^2}$$

$$\boldsymbol{E} = E\,\boldsymbol{z}\,, \quad E = \frac{J}{\sigma} = \frac{I}{\pi\sigma R^2}$$

$$\boldsymbol{H} = H(r)\,\boldsymbol{\phi}\,, \quad H(r) = \frac{I\,r}{2\pi R^2}$$

$$\boldsymbol{S} = S(r)\,\boldsymbol{r} = E\,H(r) = \frac{I^2 r}{2\pi^2 \sigma R^4}$$

Flux of S through the cable's outer surface:

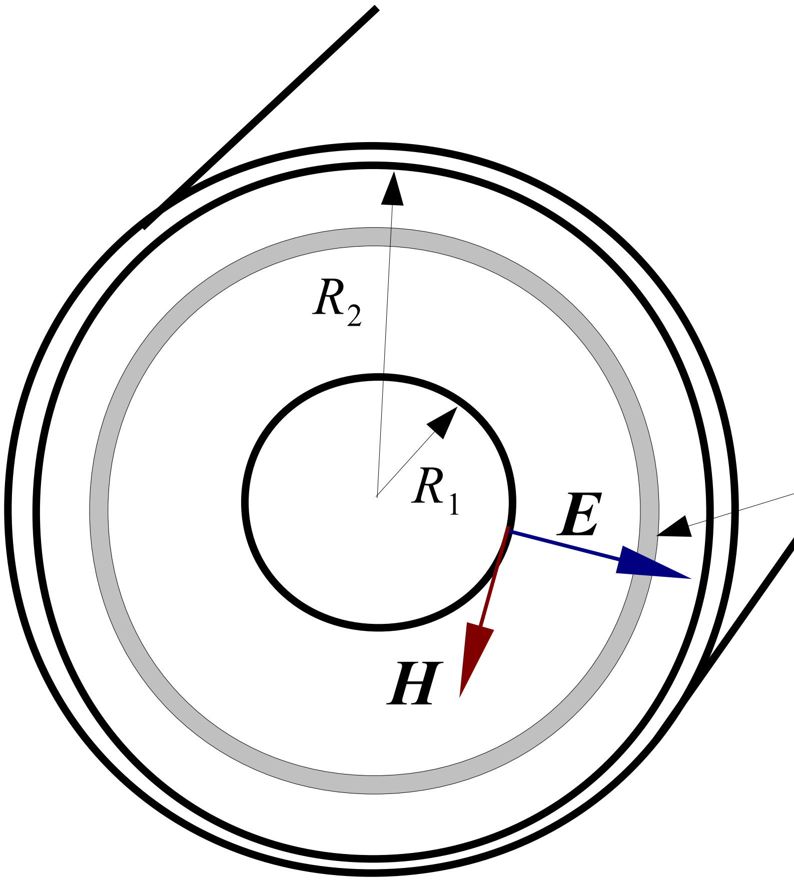
$$\iint_O \boldsymbol{S}\,dO = 2\pi R l \cdot S(R) = \frac{2\pi R l I^2 R}{2\pi^2 \sigma R^4} = I^2 \frac{l}{\sigma\pi R^2}$$

$$\frac{l}{\sigma\pi R^2}$$ – resistance of cylinder *(R,l)*

The flux of the Poyting vector through the outer surface of the cylindrical cable of length *l* is same as the ohmic losses in the cable.

Energy is radiated into metal and dissipated there as heat.

Poynting vector in the coaxial cable



The flux of the Poyting vector through cross-section of dielectric is equal to the power transmitted in the cable.

Flux of S through the cross-section of dielectric layer:

$$H_\phi = \frac{I}{2\pi r} \qquad E_r = \frac{U}{r\ln(R_2/R_1)}$$

$$S_z = \frac{UI}{2\pi r^2 \ln(R_2/R_1)}$$

$$dp = S_z 2\pi r\, dr$$

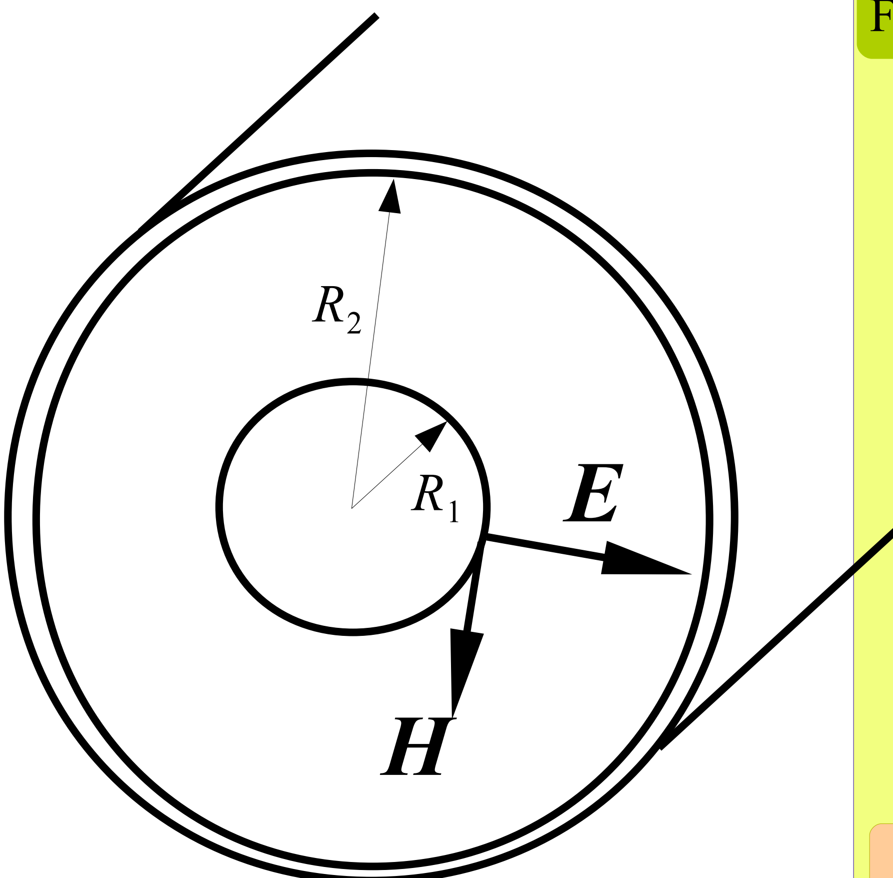
$$P = \int_{R_1}^{R_2} dp = \frac{UI}{\ln(R_2/R_1)} \int_{R_1}^{R_2} \frac{1}{r} dr =$$

$$= \frac{UI}{\ln(R_2/R_1)} \cdot \ln(R_2/R_1)$$

$$P = UI$$

Coaxial cable

(continued)



Note, that we shall obtain the same result for the outer shield: energy is radiated into metal and dissipated there.

Flux of S through the outer surface of the inner cable:

$$H_\phi = \frac{I}{2\pi R_1} \quad E_Z = \frac{J}{\sigma} = \frac{I}{\pi R_1^2} \cdot \frac{1}{\sigma}$$

$$\boldsymbol{S}(R_1) = \boldsymbol{E}(R_1) \times \boldsymbol{H}(R_1)$$

$$S = \frac{I^2}{2\pi^2 \sigma R_1^3}$$

$$\iint_O S\, dO = 2\pi R_1 l \cdot S = I^2 \cdot \frac{l}{\pi \sigma R_1^2}$$

Again the flux of the Poyting vector is same as the ohmic losses in the cable.

Energy is radiated into metal and dissipated there as heat.

Poyting vector for harmonic fields

$$\underline{\boldsymbol{S}} = \frac{1}{2}\left(\underline{\boldsymbol{E}} \times \underline{\boldsymbol{H}}^*\right) = \boldsymbol{P} + j\boldsymbol{Q}$$

$$P = Re\frac{1}{2}\oint_s \left(\underline{\boldsymbol{E}} \times \underline{\boldsymbol{H}}^*\right) dS$$

$$Q = Im\frac{1}{2}\oint_s \left(\underline{\boldsymbol{E}} \times \underline{\boldsymbol{H}}^*\right) dS$$

Power in electric circuit:

$$S = P + jQ$$

Complex Poyting vector allows one to calculate circuit parameters for harmonic signals

$$\underline{Z} = \frac{1}{I^2}\left[\frac{1}{2}\oint_s \left(\underline{\boldsymbol{E}} \times \underline{\boldsymbol{H}}^*\right) dS\right]$$

$$R = \frac{1}{I^2}\left[Re\frac{1}{2}\oint_s \left(\underline{\boldsymbol{E}} \times \underline{\boldsymbol{H}}^*\right) dS\right]$$

$$X = \frac{1}{I^2}\left[Im\frac{1}{2}\oint_s \left(\underline{\boldsymbol{E}} \times \underline{\boldsymbol{H}}^*\right) dS\right]$$

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Mechanical forces

Electric field

$$f_e = \rho \boldsymbol{E} - \frac{1}{2} E^2 \nabla \varepsilon$$

Magnetic field

$$f_m = \boldsymbol{J} \times \boldsymbol{B} - \frac{1}{2} H^2 \nabla \mu$$

Electromagnetic field

$$f_{em} = f_e + f_m + \frac{\partial p}{\partial t}$$

$$p = \varepsilon \mu \left(\boldsymbol{E} \times \boldsymbol{H} \right) = \frac{1}{c^2} \left(\boldsymbol{E} \times \boldsymbol{H} \right)$$

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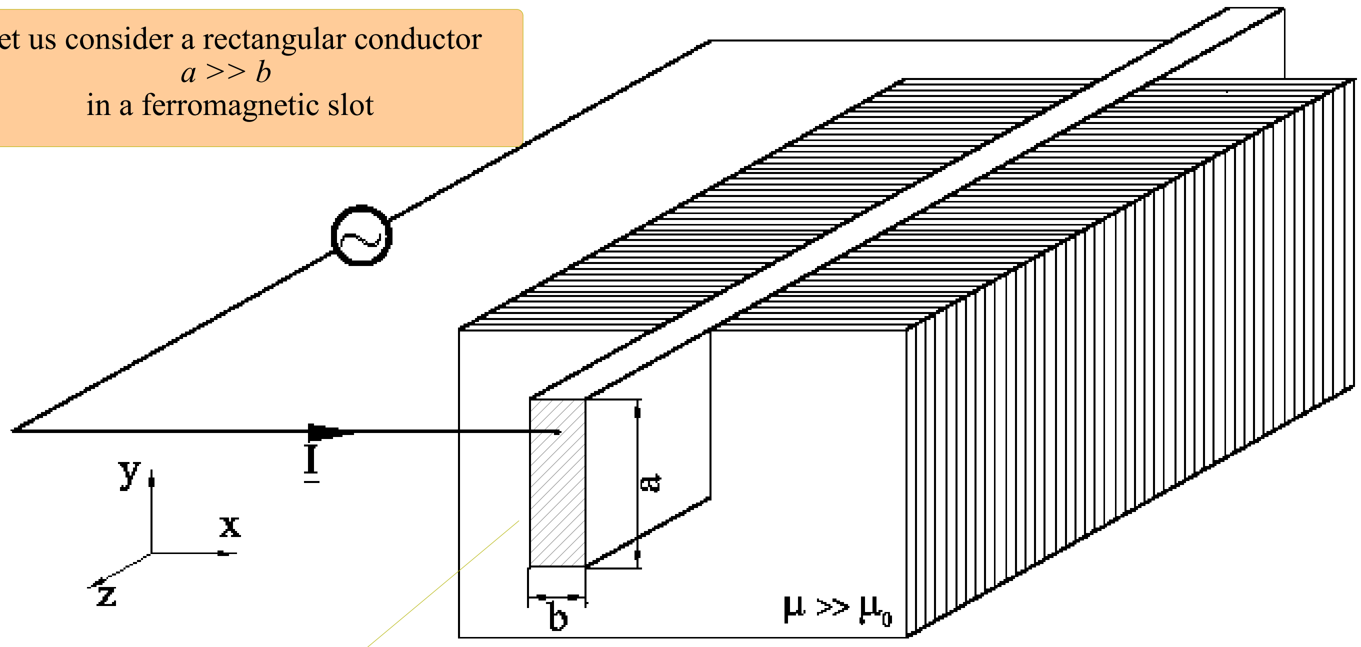
Skin effect

Let us consider a rectangular conductor
$a >> b$
in a ferromagnetic slot

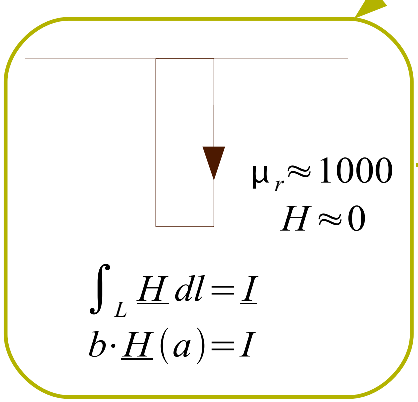


Assuming that ferromagnetic material is not saturated and that the height of the slot is much greater than its width we can use simplified analytical model:

$$\boldsymbol{H} = [\, H_x(y),\ 0,\ 0 \,]$$

$$\boldsymbol{J} = [\, 0, 0, J_z(y) \,]$$

Solution of this model shows, that current is pushed out of the slot: most of current flows near to the slot surface.



$$\begin{cases} \dfrac{\partial^2 \underline{H}(y)}{\partial y^2} - \underline{\Gamma}^2 \underline{H}(y) = 0 \\ \underline{H}(0) = 0, \\ \underline{H}(a) = \dfrac{\underline{I}}{b} \end{cases}$$

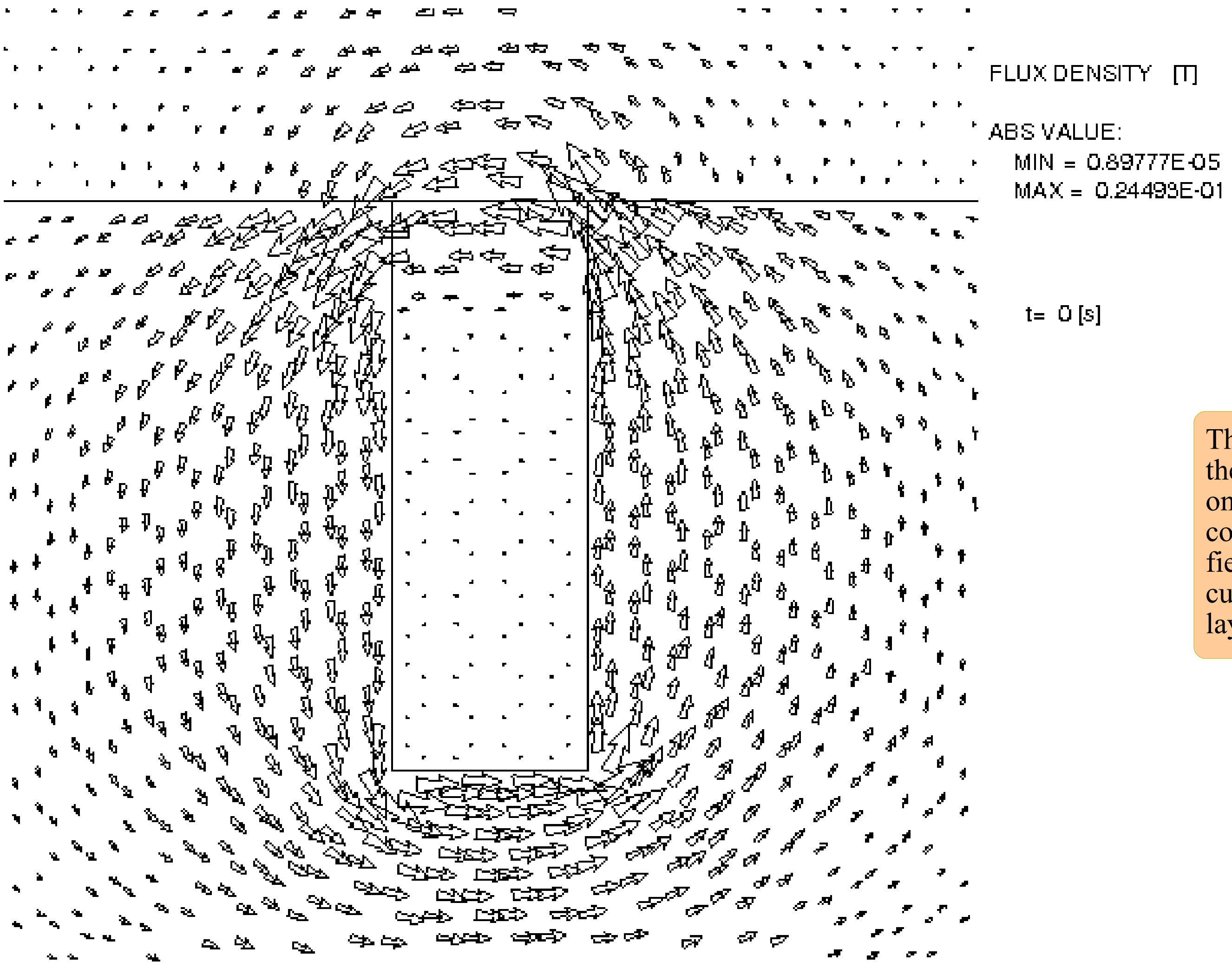
$$\underline{H}(y) = \frac{\underline{I} \sinh(\underline{\Gamma}\, y)}{b \sinh(\underline{\Gamma}\, a)}$$

$$\nabla \times \underline{H} = \underline{J}$$

$$J_z(y) = \frac{-\partial H_x}{\partial y}$$

$$\underline{J}(y) = \frac{\underline{I}\, \underline{\Gamma} \cosh(\underline{\Gamma}\, y)}{b \sinh(\underline{\Gamma}\, a)}$$

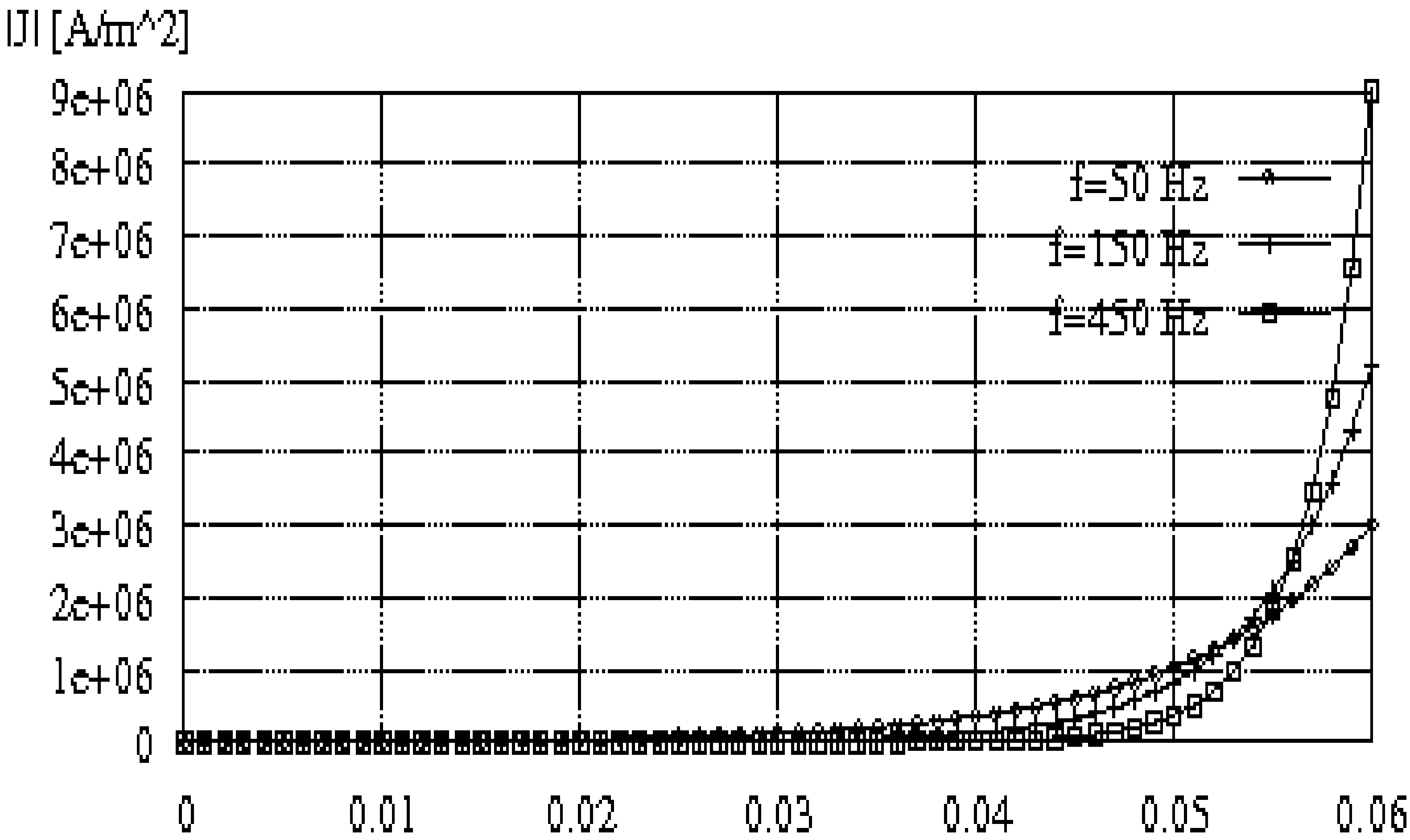
Skin effect – a numerical model



This model is much more complicated then than the analytical model shown on the previous slide, but the general conclusion is same in both cases: the field I "pushed" out of the slot and the currents flow mostly only in the surface layer of the conductor.

Skin effect at different frequencies

Increase of frequency makes skin effect stronger – at high frequencies only the very thin layer of the conductor is affected by the current.



This phenomena is used in surface hardening of material by using induction heating and by multi-frequency nondestructive testing with eddy currents.

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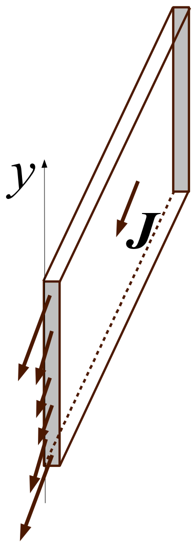
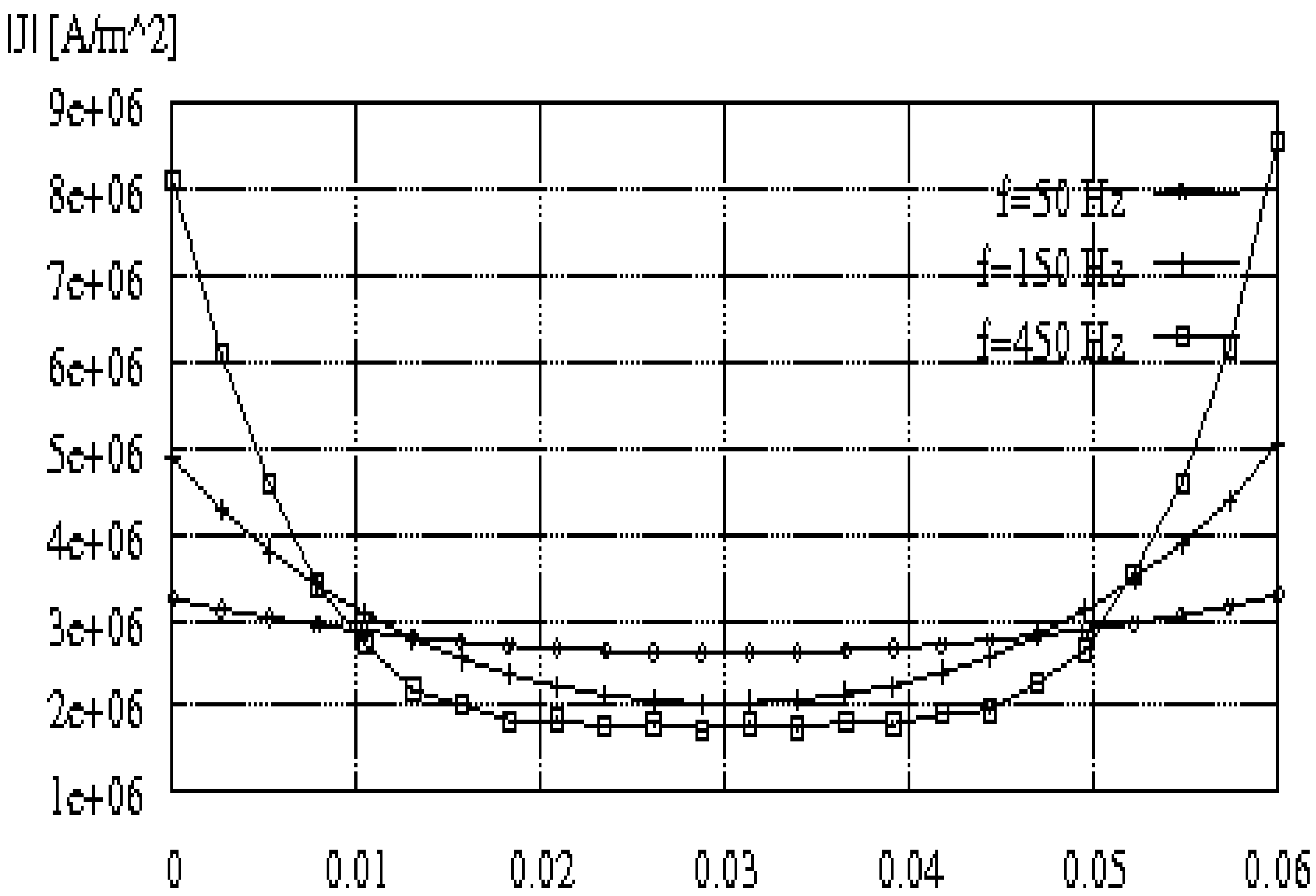
Skin effect – cable in the air

If we take cable out of the slot the effect gets symmetric: the current is pushed out of the conductor, not out of the slot. If the cable is flat (like a strip) we can observe pactically 1D effect, shown in the figure.





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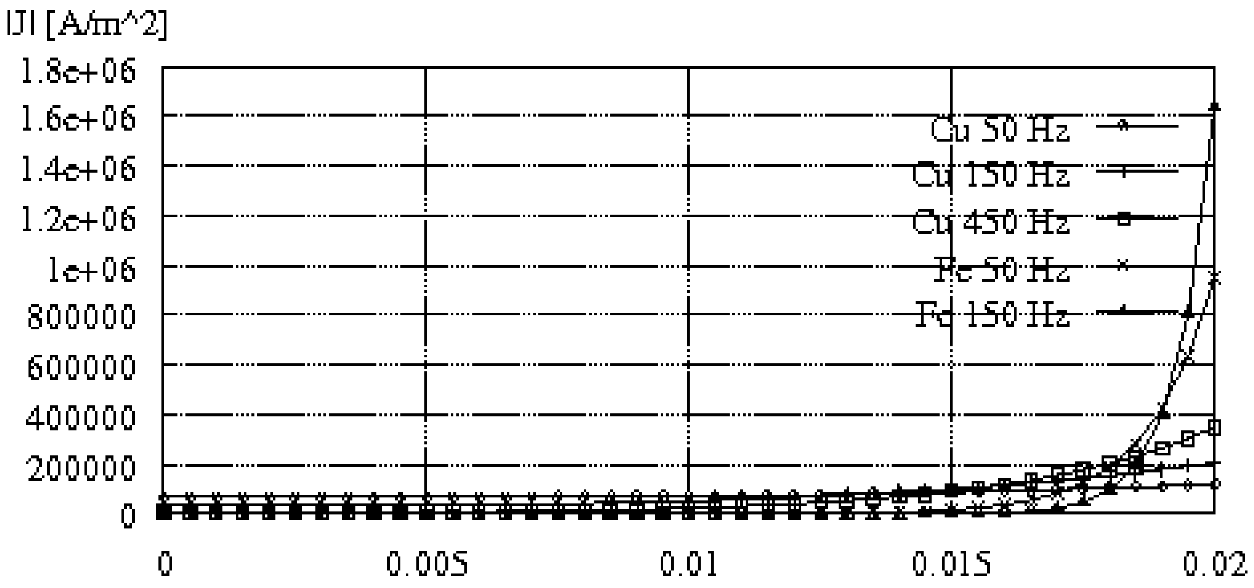
Cylindrical cable

We can observe similar effect in cylindrical cable, where we again can use relatively simple, analytical model to get approximate solution for electric or magnetic field. Current if pushed out of the center of the cable and it flows in relatively thin layer close to the surface.

$$\frac{d^2 \underline{J}}{d r^2}+\frac{1}{r}\frac{d \underline{J}}{d r}-j\omega\sigma\mu \underline{J}=0$$

Bessel function

$$\underline{J}(r)=\frac{\underline{I}\sqrt{(-j\omega\sigma\mu)}J_0(\sqrt{(-j\omega\sigma\mu)})}{2\pi R J_1(\sqrt{(-j\omega\sigma\mu)})}$$



Strong skin effect – simple model

When can we use it?

Conductor thickness >> skin depth

How?

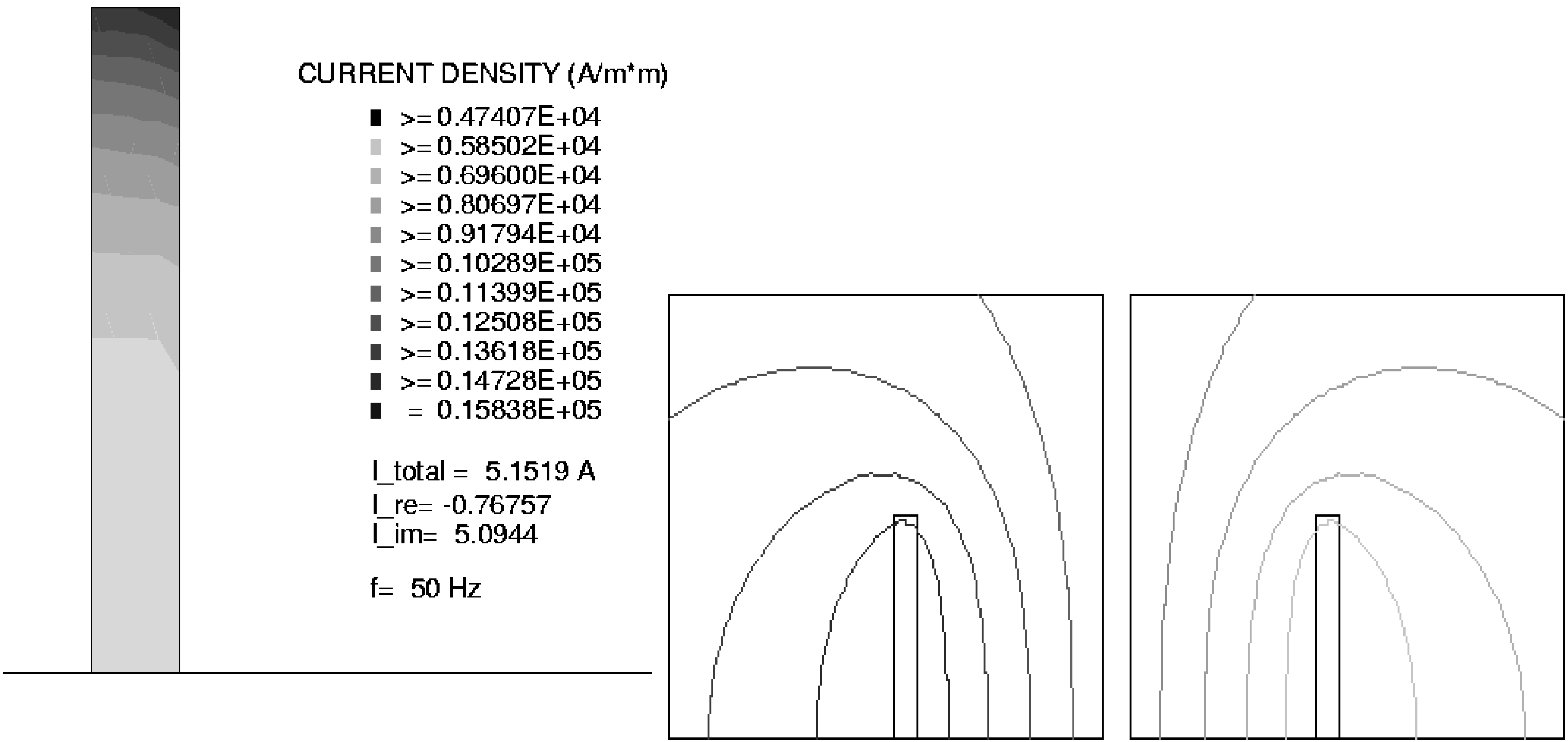
We consider EM wave penetrating the conductor

Skin depth: $d = \sqrt{\dfrac{2}{\omega \sigma \mu}}$

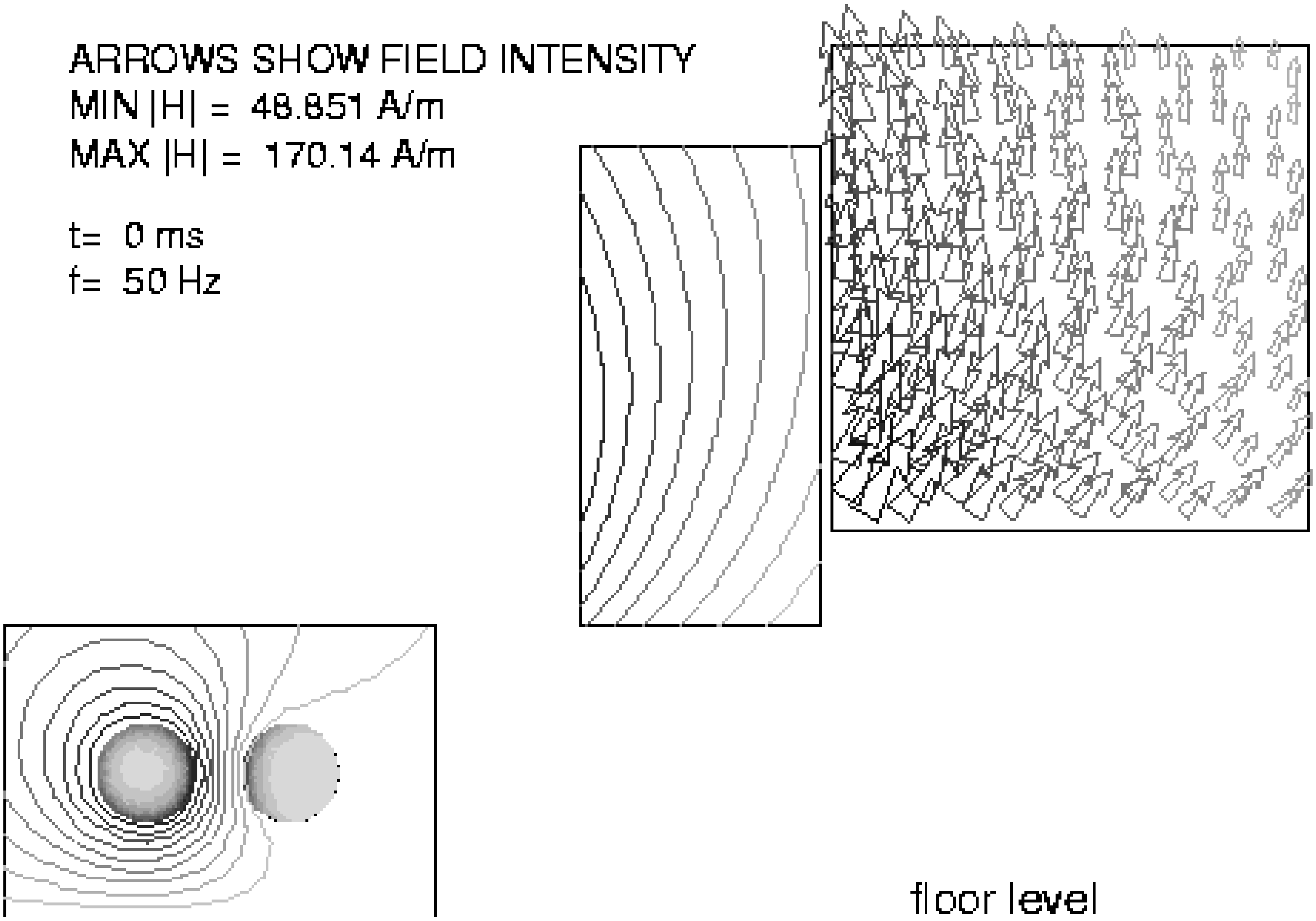
$$\underline{H}(r) = \underline{H}_s e^{-\underline{\Gamma}(R-r)} \qquad \underline{H}_s = \frac{\underline{I}}{2\pi R}$$

$$\underline{J}(r) = \underline{H}_s \sqrt{\omega \sigma \mu}\, e^{j\pi/4} e^{-\underline{\Gamma}(R-r)}$$

Proximity effect



Electromagnetic screen



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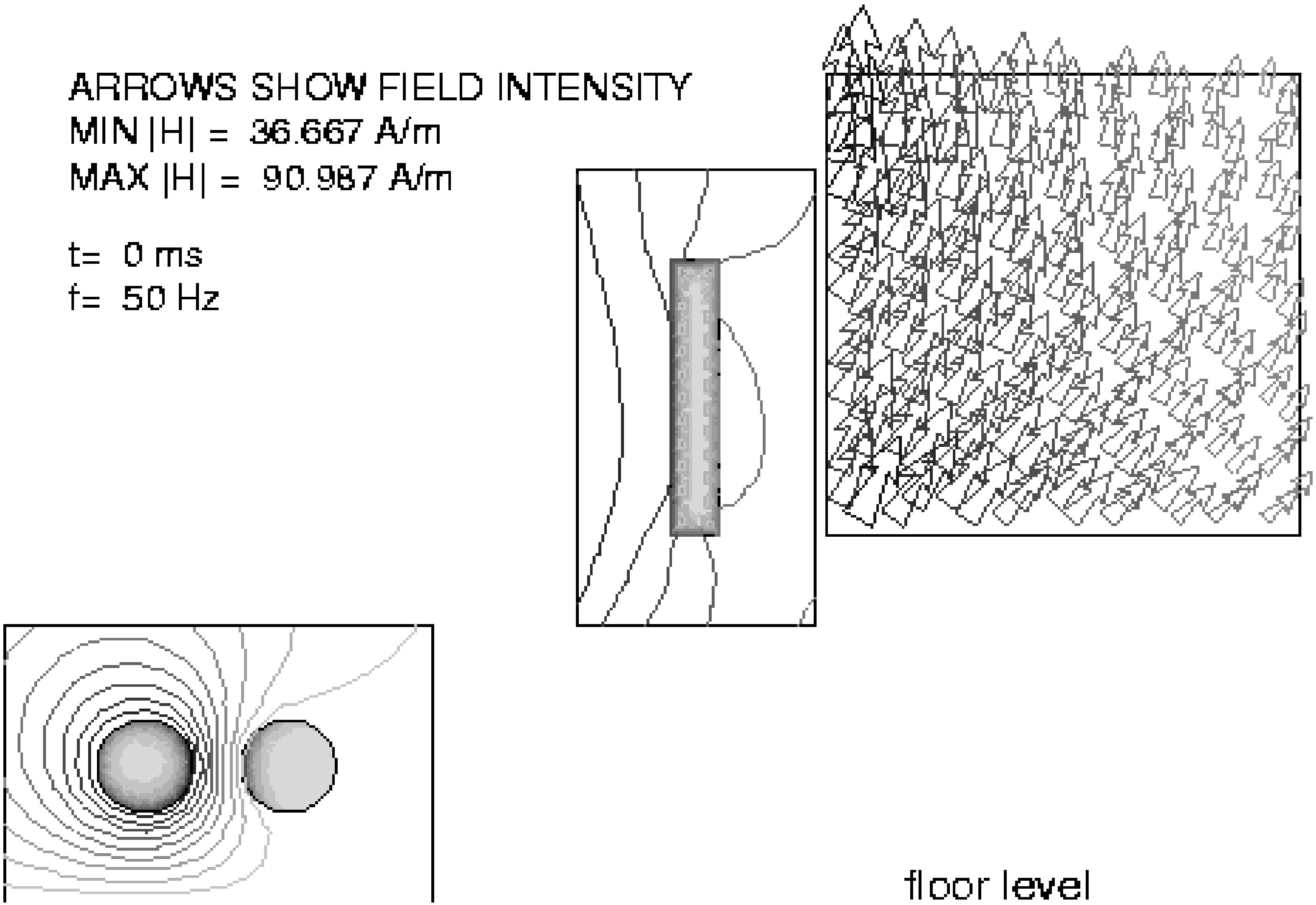
Electromagnetic screen
(continued)



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