# Electromagnetic Fields

*Lecture 3*

# *Electrostatics 1*

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# Coulomb's law

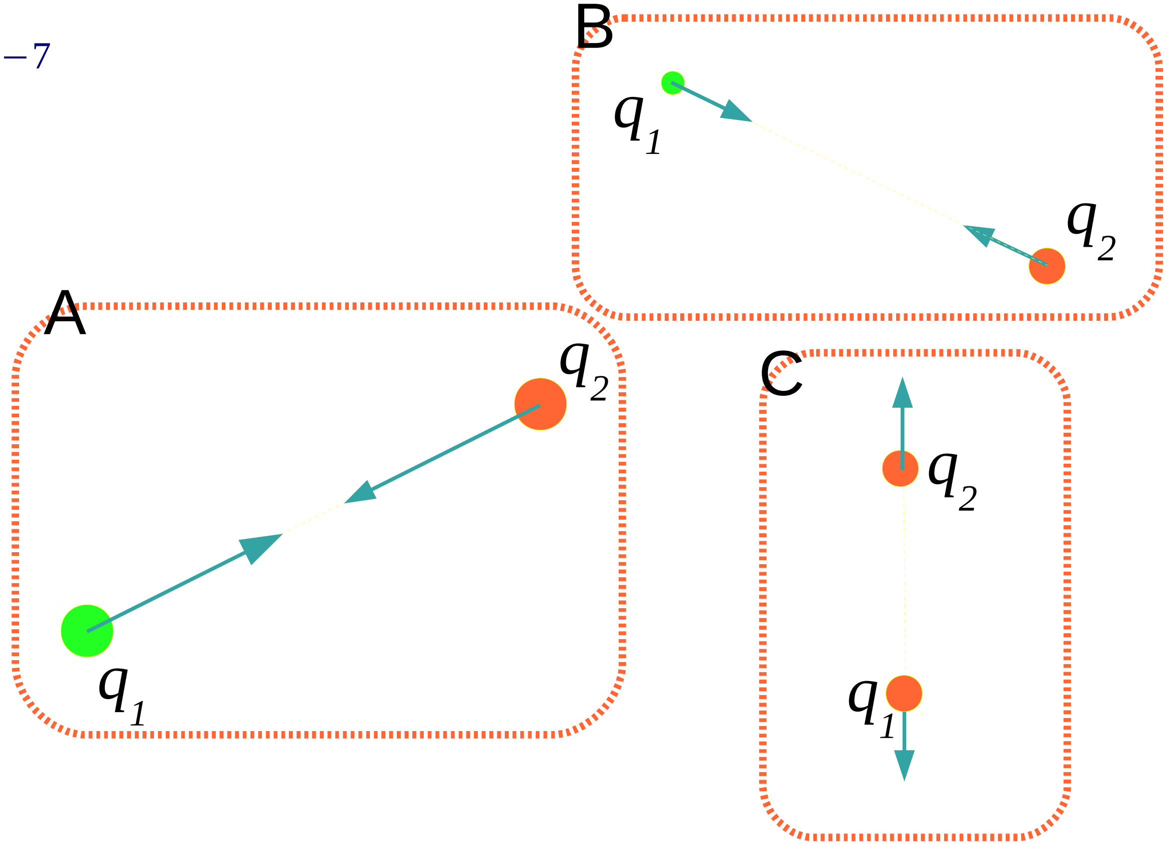
$$\boldsymbol{F}=k\frac{q_1 q_2}{r^2}\mathbf{1}_{q_1 q_2}$$

$$k=\frac{1}{4\pi\varepsilon_0}=c^2\frac{\mu_0}{4\pi}=c^2\cdot 10^{-7}$$

$$k=8.988\times 10^{-9}$$

$$\varepsilon_0=8.854\times 10^{-12}$$

**Historical note:**
1767: John Priestley - suggestion
1771: Henry Cavendish - experiment
1785: Charles Augustin de Coulomb - publication



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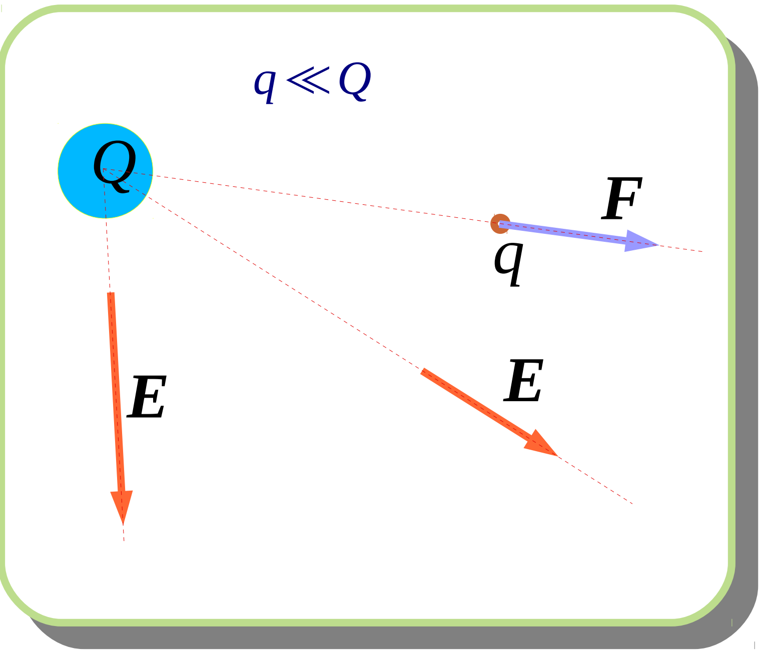
# Electric Field of single charge




$$\boldsymbol{F} = k \frac{Q}{r^2} \mathbf{1}_{Qq} \cdot q = q\,\boldsymbol{E}$$

$$\boldsymbol{E} = \frac{\boldsymbol{F}}{q}$$

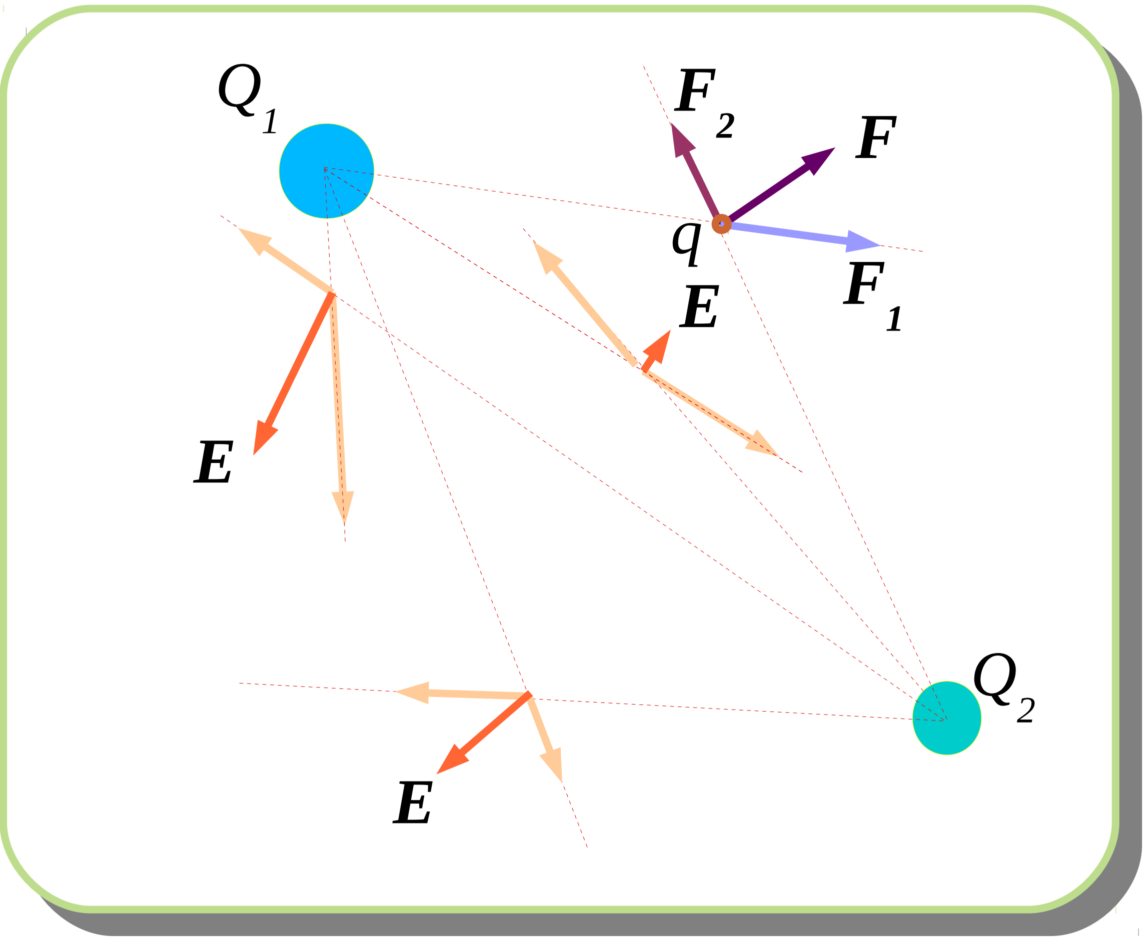
The electric field is defined as the force per unit charge.
(As if it would act on a *stationary point* charge)

$$\boldsymbol{E} = \lim_{q \to 0} \frac{\boldsymbol{F}}{q}$$

E due to a single discrete (point) charge Q:

$$\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \mathbf{1}_r$$

# Electric Field of many Charges




$$\boldsymbol{F}_1 = k\frac{Q_1}{r_i^2}\mathbf{1}_{Q_1 q}\cdot q = q\,\boldsymbol{E}_1$$

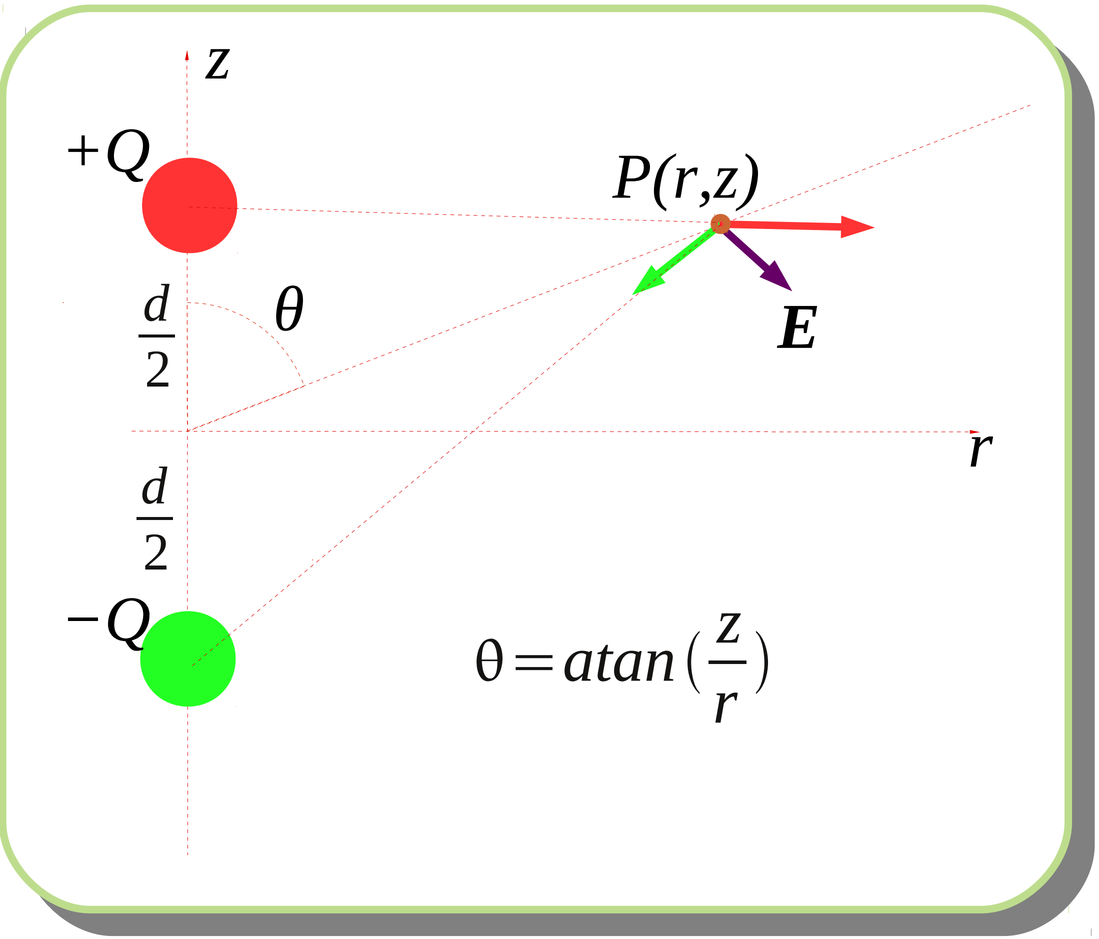
$$\boldsymbol{F}_2 = k\frac{Q_2}{r_i^2}\mathbf{1}_{Q_2 q}\cdot q = q\,\boldsymbol{E}_2$$

$$\boldsymbol{F} = k\cdot q\cdot\sum_i \frac{Q_i}{r_i^2}\mathbf{1}_{Q_i q} = q\,\boldsymbol{E}$$

***E*** due to set of discrete (point) charges $Q_1 \ldots Q_n$:

$$\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{Q_i}{r_i^2}\mathbf{1}_{r_i}$$

# Electric Dipole




$$\boldsymbol{E}=\boldsymbol{E}_{+}+\boldsymbol{E}_{-}$$

In cylindrical coordinates:

$$\boldsymbol{E}_{+}=\frac{kQ}{r_{+}^{2}}\left[\frac{r}{r_{+}},\frac{z-d/2}{r_{+}}\right]$$

$$r_{+}=\sqrt{r^{2}+\left(z-\frac{d}{2}\right)^{2}}$$

$$\boldsymbol{E}_{-}=-\frac{kQ}{r_{-}^{2}}\left[\frac{r}{r_{-}},\frac{z+d/2}{r_{-}}\right]$$

$$r_{-}=\sqrt{r^{2}+\left(z+\frac{d}{2}\right)^{2}}$$

Far ($r >> d$) field of dipole in spherical coordinates:

$$\boldsymbol{E}=\frac{dQ}{4\pi\varepsilon_{0}}\left[2\cos(\theta),\sin(\theta)\right]$$

Dipole momentum $p = d\,Q$



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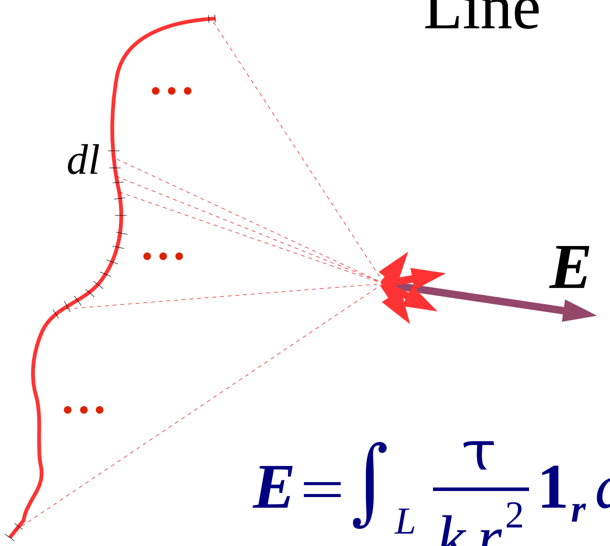
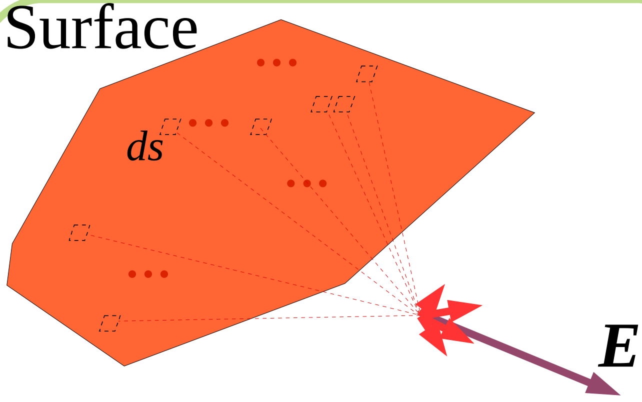
# "Continuous" Charge

Line



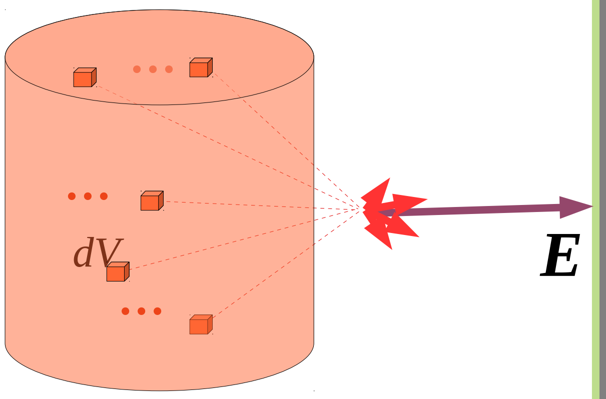

$$\boldsymbol{E} = \int_L \frac{\tau}{k\,r^2} \mathbf{1}_r \, dl$$

Surface



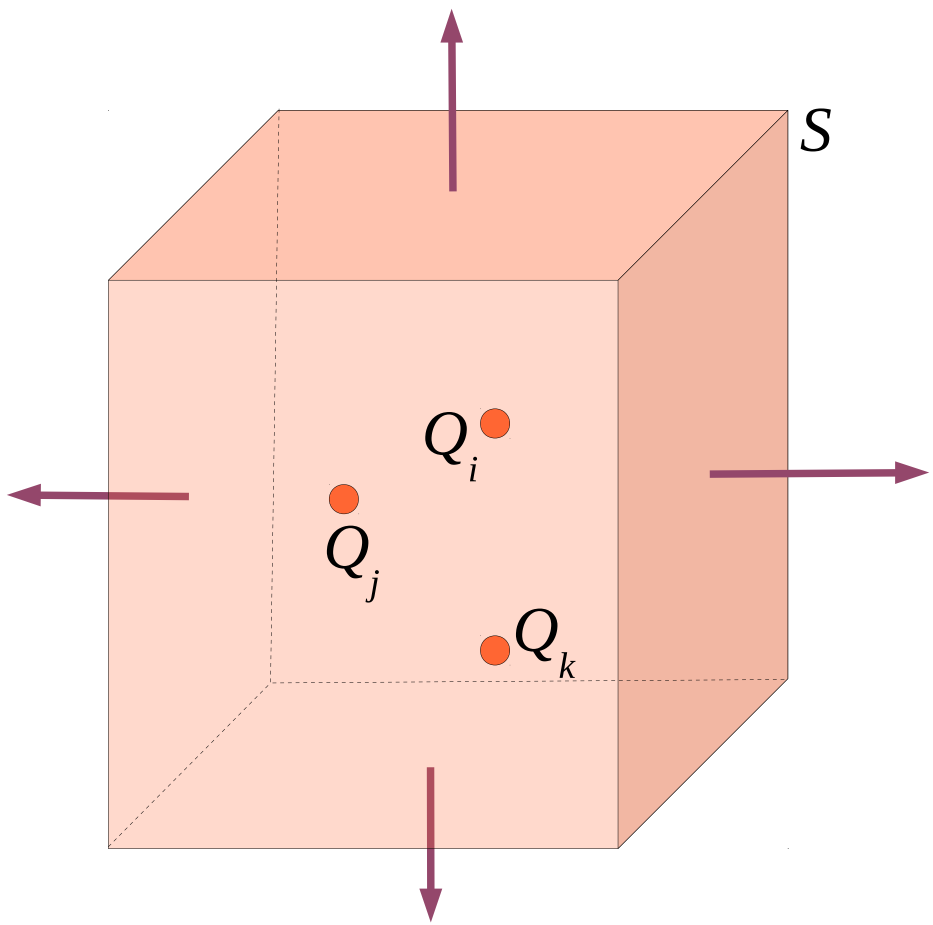

$$\boldsymbol{E} = \iint_S \frac{\sigma}{k\,r^2} \mathbf{1}_r \, ds$$

Volume




$$\boldsymbol{E} = \iiint_V \frac{\rho}{k\,r^2} \mathbf{1}_r \, dV$$

# Gauss's law




$$\oiint_S \boldsymbol{D}\, d\boldsymbol{s} = \sum Q$$

$$\oiint_S \boldsymbol{E}\, d\boldsymbol{s} = \frac{\sum Q}{\varepsilon_0}$$

$$\nabla \cdot \boldsymbol{D} = \rho$$

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$$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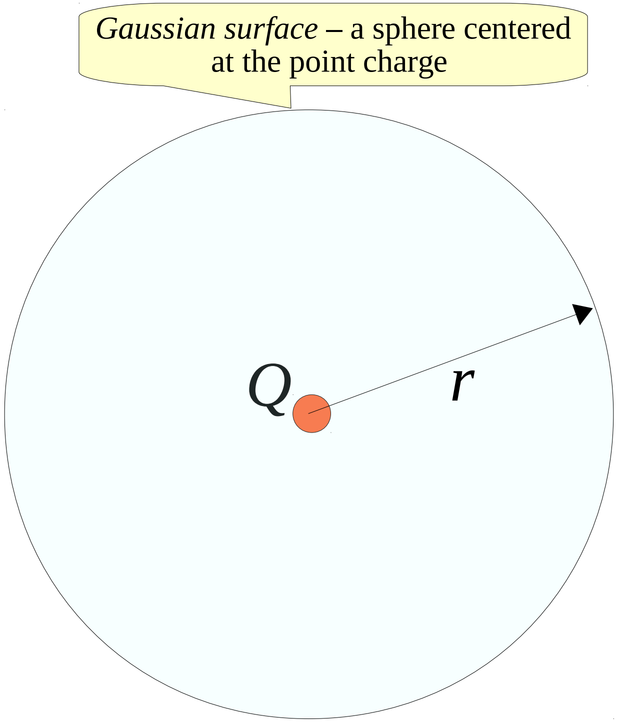
# Application of Gauss's law




Gauss's law can be used to calculate field due to specially distributed charges. If there is some sort of symmetry in the charge distribution, then knowledge of the total flux allows deduction of the field at every point.

$$\oint\!\!\oint_S \boldsymbol{D}\, d\boldsymbol{s} = Q$$

Symmetry: ***D*** has only one component (normal to sphere), which is constant on the sphere

$$4\pi r^2 \boldsymbol{D} = Q$$

$$\boldsymbol{D} = \frac{Q}{4\pi r^2} \qquad \boldsymbol{E} = \frac{Q}{4\pi \varepsilon_0 r^2}$$



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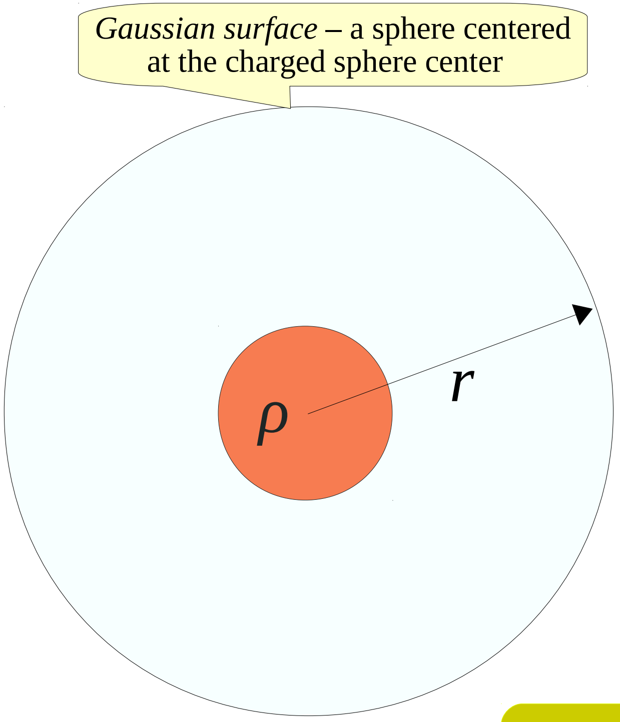
# Example: field of charged sphere




$$\oint\!\!\oint_S \mathbf{D}\, d\mathbf{s} = Q$$

Symmetry: ***D*** has only one component (normal to sphere), which is constant on the sphere

$$4\pi r^2 D = Q$$

$$\mathbf{D} = \frac{Q}{4\pi r^2} \qquad \mathbf{E} = \frac{Q}{4\pi \varepsilon_0 r^2}$$

An external observer can not determine the source of electric field with spherical symmetry:

- it can be a point charge
- it can be any volume or surface distribution with spherical symmetry (shell, sphere,...)



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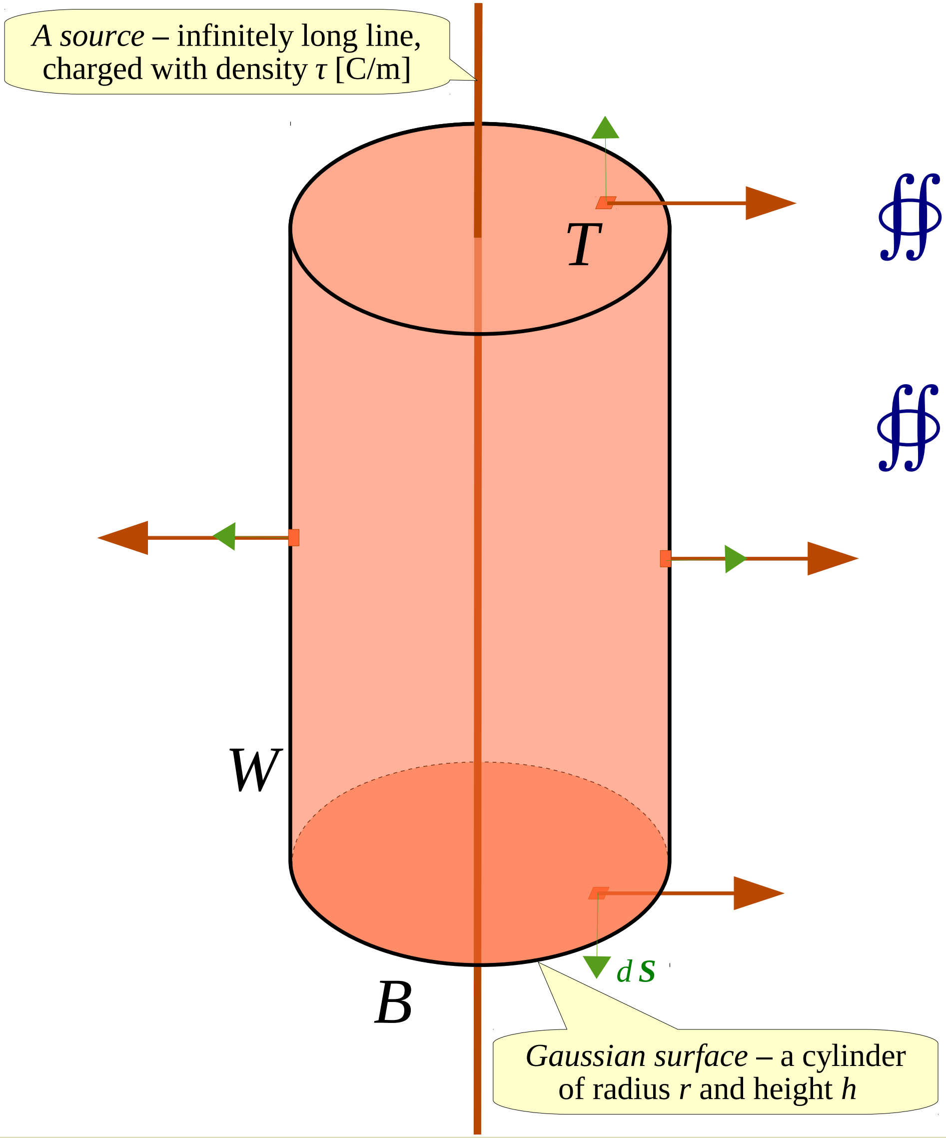
# Example: field of charged line




$$\oint\!\!\oint_S \mathbf{D}\, d\mathbf{s} = Q = h \cdot \tau$$

$$\oint\!\!\oint_S \mathbf{D}\, d\mathbf{s} = \oint_W \mathbf{D}\, d\mathbf{s} + \oint_T \mathbf{D}\, d\mathbf{s} + \oint_B \mathbf{D}\, d\mathbf{s} =$$

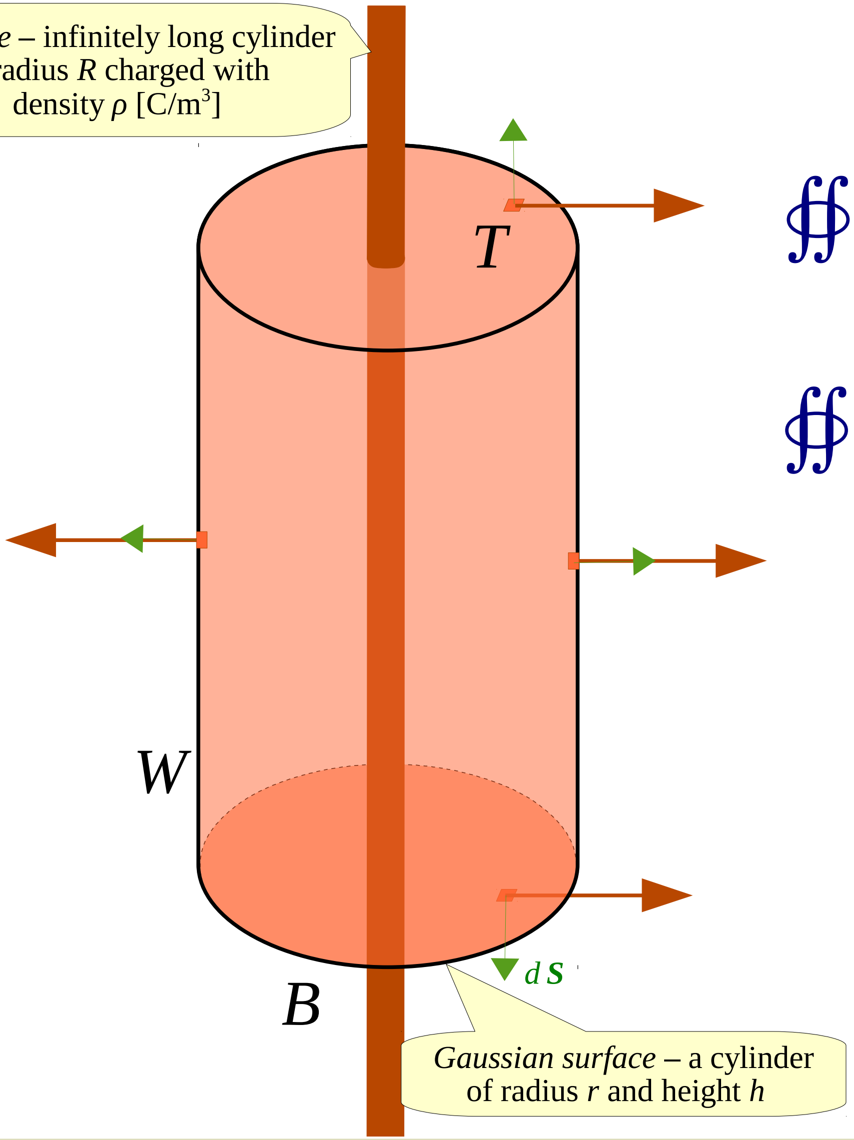
$$= \oint_W \mathbf{D}\, d\mathbf{s} + 0 + 0 =$$

$$= 2\pi r h \mathbf{D}$$

$$\not{h} \cdot \tau = 2\pi r \not{h} \mathbf{D}$$

$$\mathbf{D} = \frac{\tau}{2\pi r}$$

# Example: field of charged cylinder




$$\oiint_S \boldsymbol{D}\,d\boldsymbol{s} = Q = \pi R^2 h \cdot \rho$$

$$\oiint_S \boldsymbol{D}\,d\boldsymbol{s} = \oint_W \boldsymbol{D}\,d\boldsymbol{s} + \oint_T \boldsymbol{D}\,d\boldsymbol{s} + \oint_B \boldsymbol{D}\,d\boldsymbol{s} =$$

$$= \oint_W \boldsymbol{D}\,d\boldsymbol{s} + 0 + 0 =$$

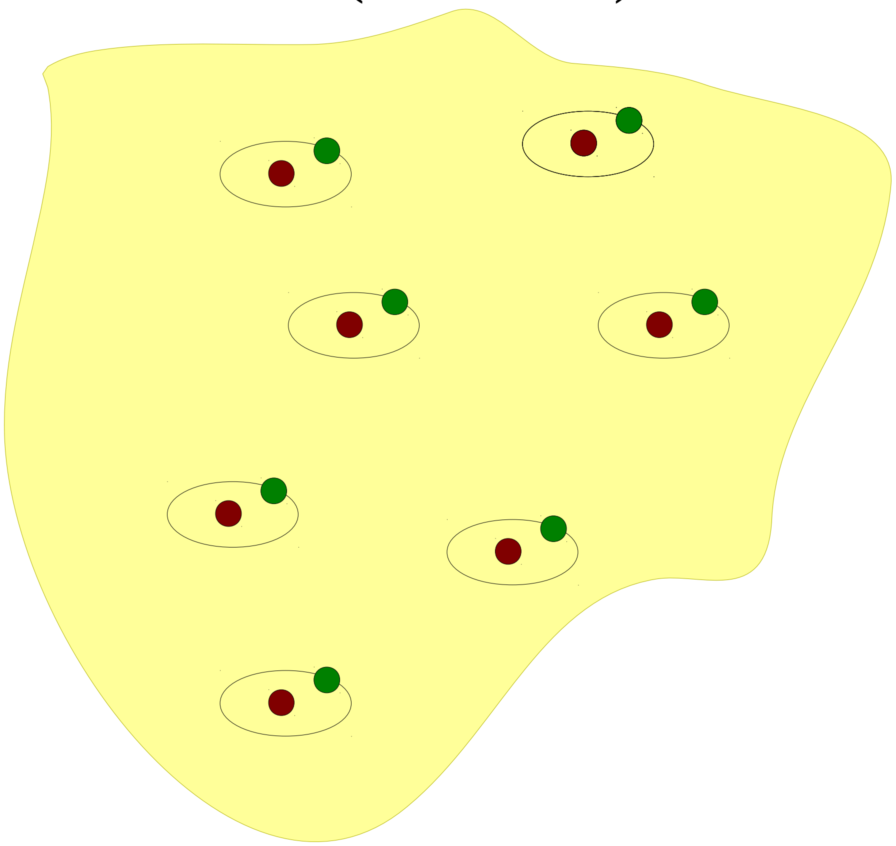
$$= 2\pi r h \boldsymbol{D}$$

$$\pi R^2 h \cdot \rho = 2\pi r h \boldsymbol{D}$$

$$\boldsymbol{D} = \frac{R^2 \rho}{2r} = \frac{\tau'}{2\pi r} \qquad \tau' = \pi R^2 \rho$$

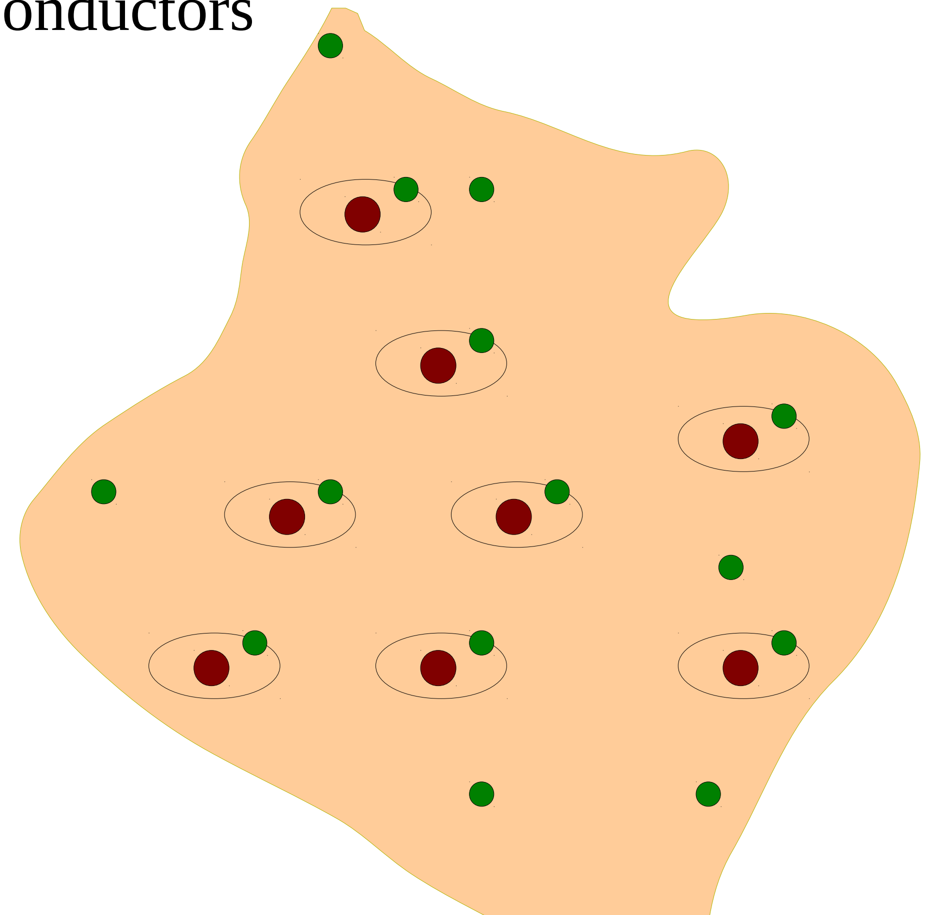
# Electricity and matter

Dielectrics (insulators)



Positive and negative charges are bounded in particles (or atoms)

Conductors



Positive and negative charges may separate forming movable charges



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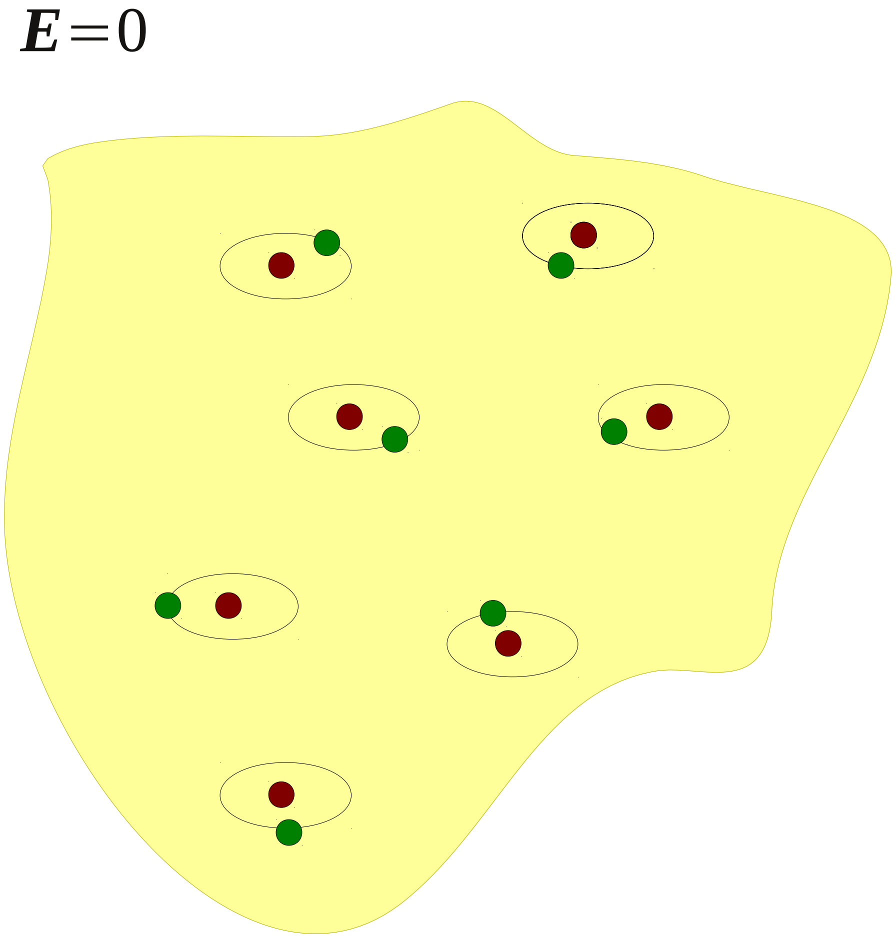
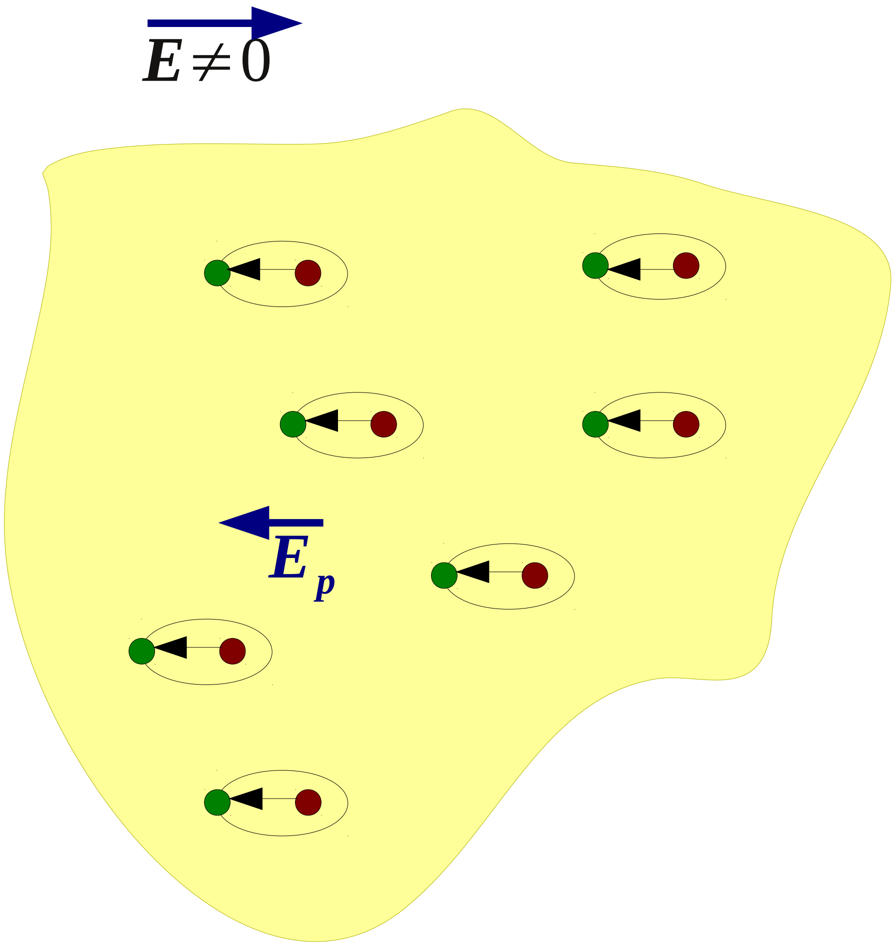
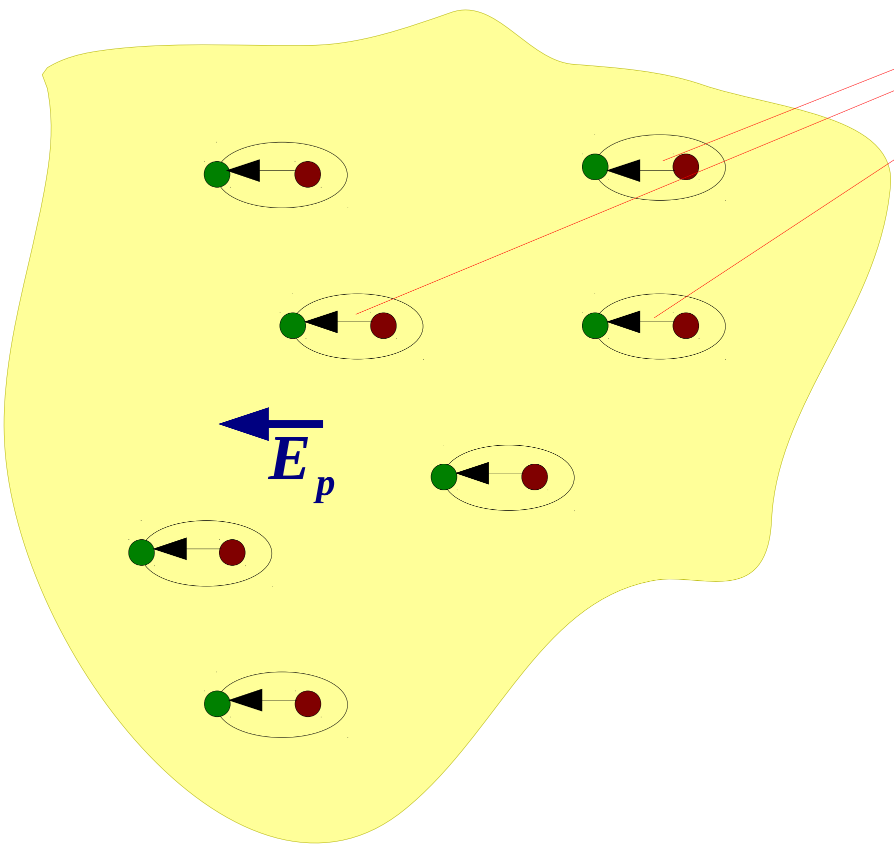
# Dielectrics: polarization



# Dielectrics: polarization




$$\boldsymbol{P} = \lim_{v \to 0} \frac{\sum p_i}{v}$$

$$\boldsymbol{D} = \varepsilon_0 E + P$$

$$\boldsymbol{D} = \varepsilon E$$

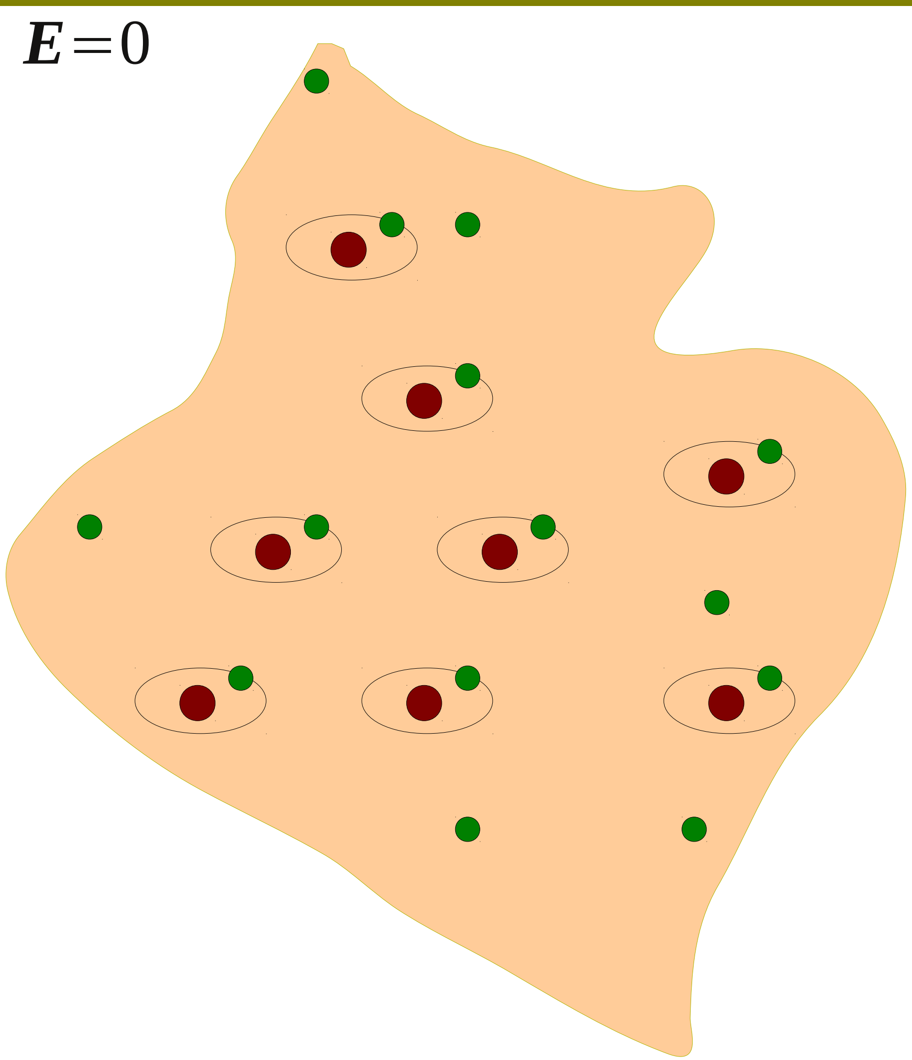
$$\varepsilon = \varepsilon_r \varepsilon_0 = (1 + \kappa) \varepsilon_0$$

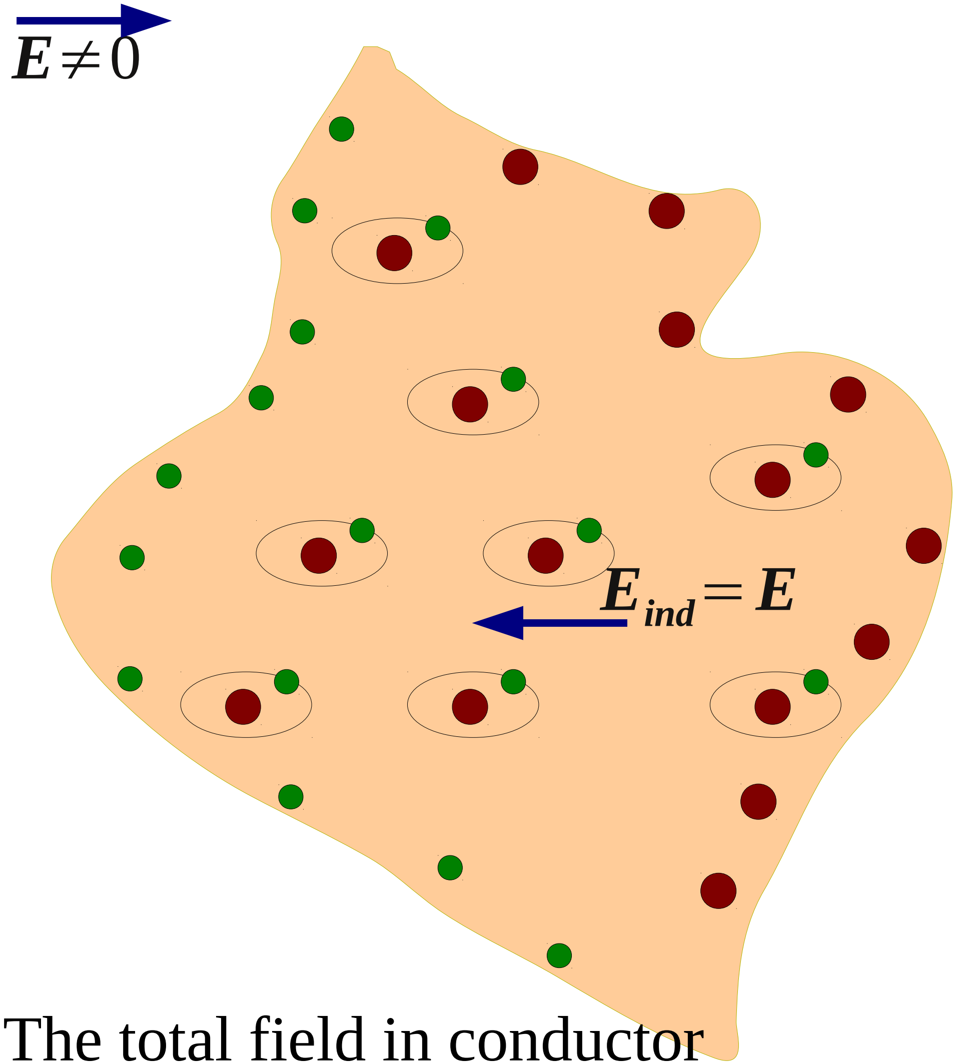
Electric susceptibility

$$\varepsilon_r \in \langle 1, 150 \rangle$$

Conjugated polymers up to $10^5$

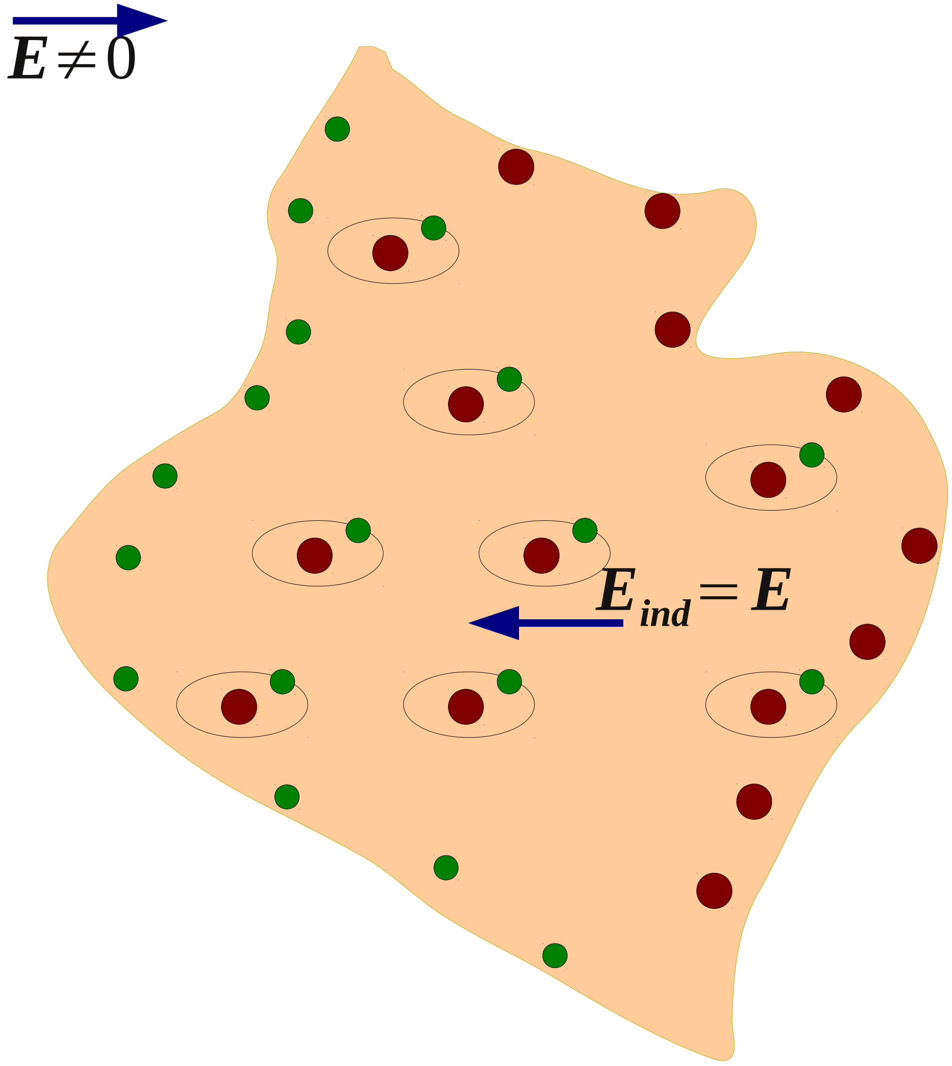
# Electric induction in conductors




The total field in conductor
is equal to ZERO!

# Electric induction in conductors




$$\nabla \cdot D = \rho \qquad D = \varepsilon E$$

$$\nabla \cdot J = -\frac{d\rho}{dt} \qquad J = \sigma E$$

$$\nabla \cdot (\sigma E) = -\nabla \cdot \left( \varepsilon \frac{dE}{dt} \right)$$

$$\nabla \cdot \left( \sigma E + \varepsilon \frac{dE}{dt} \right) = 0$$

$$E + \frac{\varepsilon}{\sigma} \frac{dE}{dt} = 0$$

$$E = E_0 \cdot e^{-\frac{t}{\tau}}, \qquad \tau = \frac{\varepsilon}{\sigma}$$

For Cu ($\varepsilon$=8.885e-12, $\sigma$=57e6) $\tau$=0.155e-18 s



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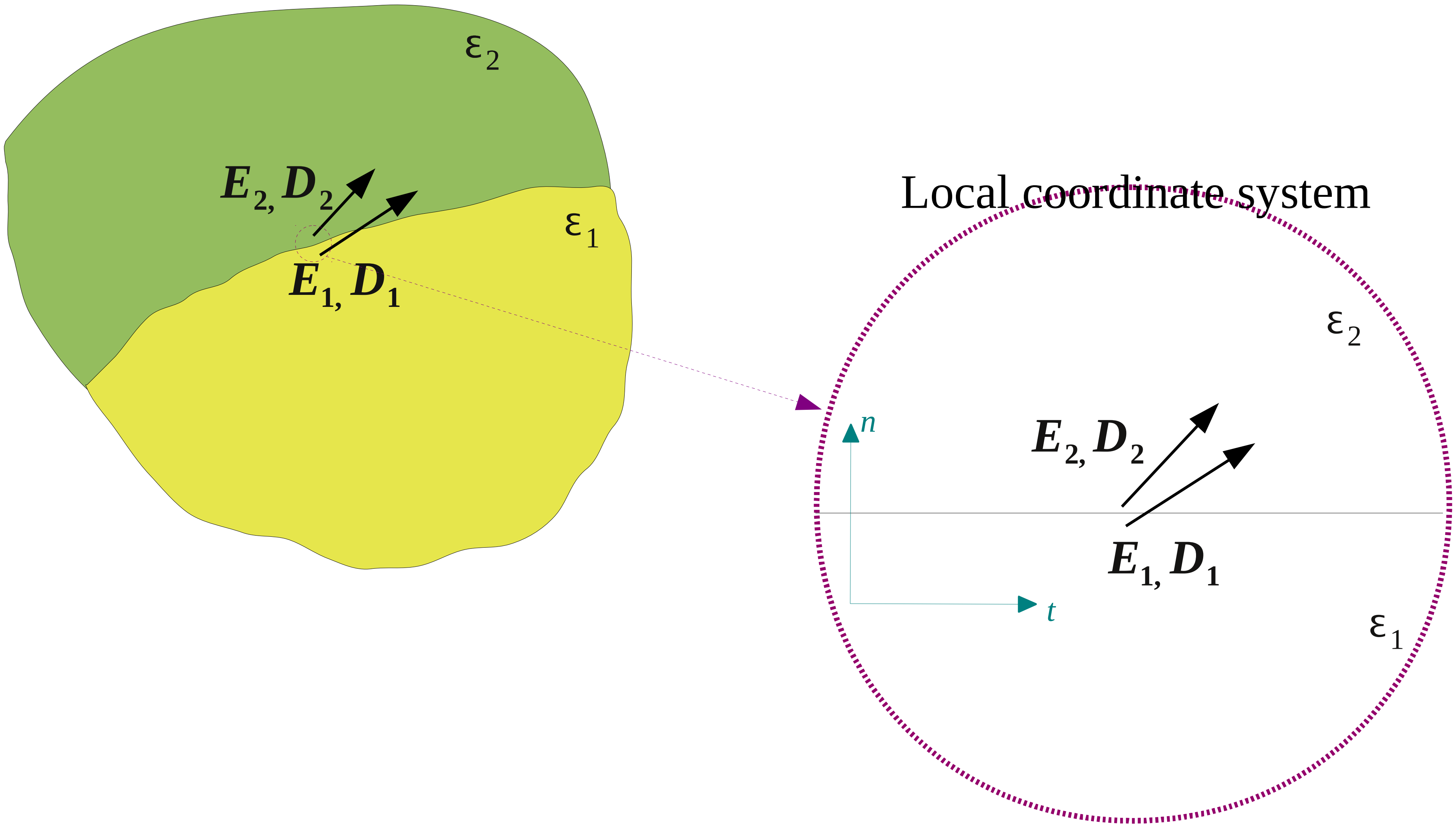
# Interface conditions



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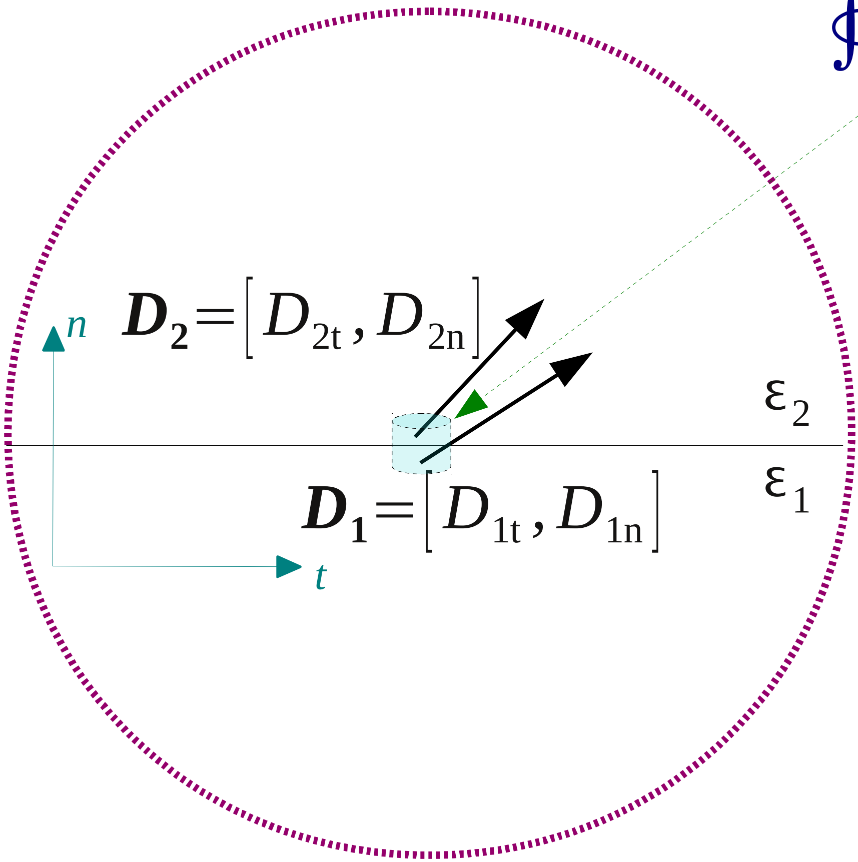
# Interface conditions for ***D***




$$\oint\!\!\oint_S \boldsymbol{D}\, d\,\boldsymbol{s} = \sum Q$$

$$\oint\!\!\oint_S \boldsymbol{D}\, d\,\boldsymbol{s} = \int_{top} D_{2n}\, ds + \int_{side2} D_{2t}\, ds + \int_{side1} D_{1t}\, ds - \int_{bottom} D_{1n}\, ds$$

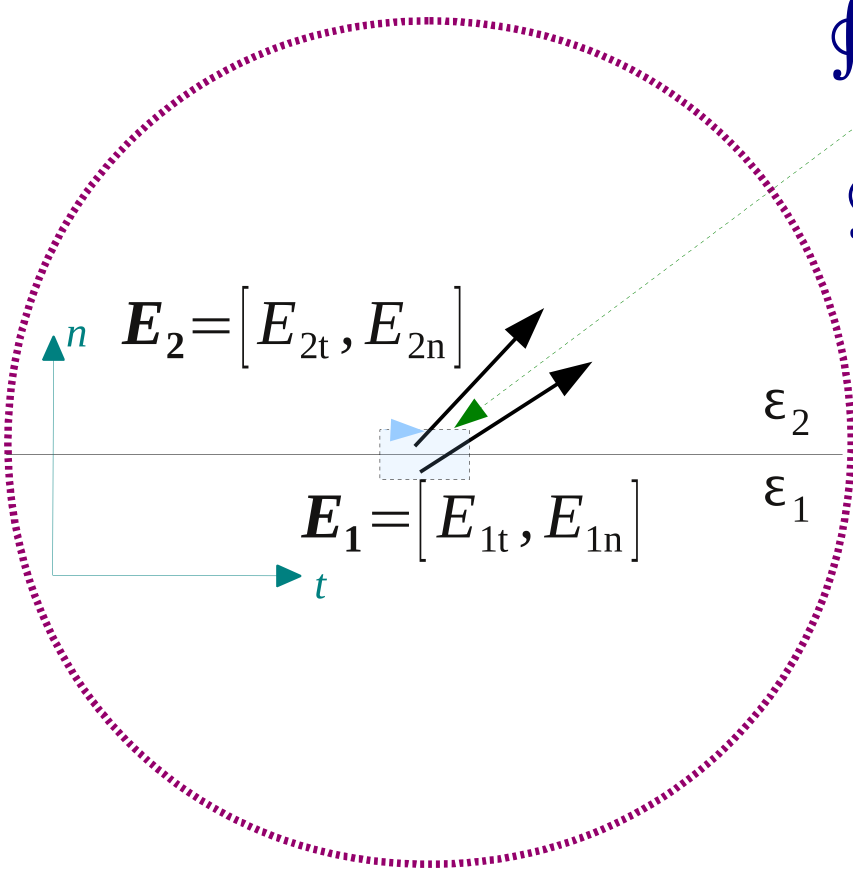
$$\int_{side2} D_{2t}\, ds = 0 \qquad \int_{side1} D_{1t}\, ds = 0$$

$$\oint\!\!\oint_S \boldsymbol{D}\, d\,\boldsymbol{s} = \int_{top} D_{2n}\, ds - \int_{bottom} D_{1n}\, ds \simeq \pi r^2 (D_{2n} - D_{1n})$$

$$\sum Q \simeq \pi r^2 \tau$$

$$D_{2n} - D_{1n} = \tau$$

# Interface conditions for *E*




$$\oint_L \boldsymbol{E}\, d\boldsymbol{l} = 0$$

$$\oint_L \boldsymbol{E}\, d\boldsymbol{l} = \int_{top} E_{2t}\, dt - \int_{bottom} E_{1t}\, dt - $$
$$+ \int_{right2} E_{2n}\, dn - \int_{right1} E_{1n}\, dn + $$
$$+ \int_{left1} E_{1n}\, dn + \int_{left2} E_{2n}\, dn$$

$$\oint_L \boldsymbol{E}\, d\boldsymbol{l} = \int_{top} E_{2t}\, dt - \int_{bottom} E_{1t}\, dt = 0$$

$$E_{2t} - E_{1t} = 0$$

# Maxwell Equations revisited

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$$
$$\nabla \times \boldsymbol{E} = \frac{-\partial \boldsymbol{B}}{\partial t}$$
$$\nabla \cdot \boldsymbol{D} = \rho$$
$$\nabla \cdot \boldsymbol{B} = 0$$
$$\boldsymbol{D} = \varepsilon \boldsymbol{E}$$
$$\boldsymbol{J} = \sigma \boldsymbol{E}$$
$$\boldsymbol{B} = \mu \boldsymbol{H}$$

We are interested in phenomena arisen from stationary or *very* slow moving charges.

$$\frac{\partial \boldsymbol{B}}{\partial t} = 0, \qquad \frac{\partial \boldsymbol{D}}{\partial t} = 0$$

Only electric field of interest.
There are no charge sources.

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}$$
$$\nabla \times \boldsymbol{E} = 0$$
$$\nabla \cdot \boldsymbol{D} = \rho$$
$$\nabla \cdot \boldsymbol{B} = 0$$
$$\boldsymbol{D} = \varepsilon \boldsymbol{E}$$
$$\boldsymbol{J} = \sigma \boldsymbol{E}$$
$$\boldsymbol{B} = \mu \boldsymbol{H}$$

We may use scalar, not vector !!

$$\boldsymbol{E} = \nabla \varphi$$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# Scalar electric potential

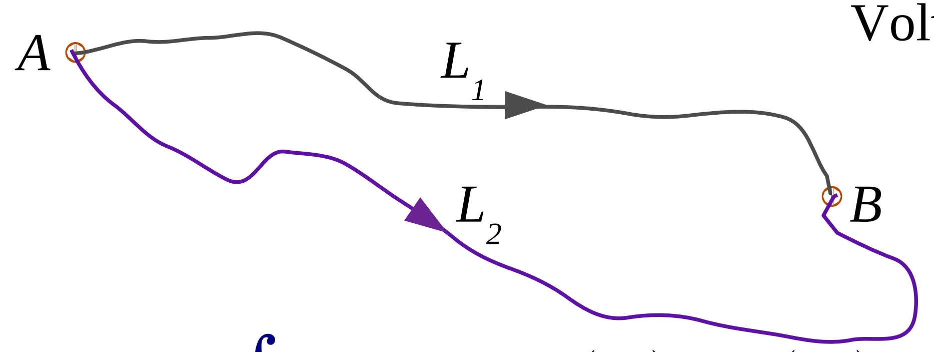
$$\boldsymbol{E} = -\nabla \varphi$$

Electric potential can be given by the line integral of ***E*:**

$$\varphi(P) = -\int_{ref}^{P} \boldsymbol{E}\, d\boldsymbol{l}$$

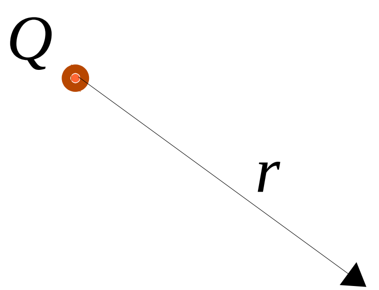
where *ref* is some point at which *φ = 0.*

## Voltage




$$\int_{L_1} \boldsymbol{E}\, d\boldsymbol{L} = \int_{L_2} \boldsymbol{E}\, d\boldsymbol{L} = \varphi(B) - \varphi(A)$$

## Coulomb potential




$$\varphi(r) = k\frac{Q}{r}$$

## Potential energy

Work in electric field

$$dW = \boldsymbol{F} \cdot dl = q\,\boldsymbol{E} \cdot dl$$

$$W = \int_{L} \boldsymbol{F} \cdot d\boldsymbol{L} = q\int_{L} \boldsymbol{E} \cdot d\boldsymbol{L} = q\,U_{AB}$$

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# Poisson's & Laplace's equations

$\nabla \times \boldsymbol{E} = 0$ → $\boldsymbol{E} = -\nabla \varphi$

$\boldsymbol{D} = \varepsilon \boldsymbol{E}$ → $\boldsymbol{D} = -\varepsilon \nabla \varphi$

$\nabla \cdot \boldsymbol{D} = \rho$ → $\nabla \cdot (\varepsilon \nabla \varphi) = -\rho$

$\varepsilon = const$

$$\nabla \cdot \nabla \varphi = -\frac{\rho}{\varepsilon}$$

Poisson's equation

$\rho = 0$

$$\nabla \cdot \nabla \varphi = 0$$

Laplace's equation

Math: Laplacian (Laplace's operator)

$$\nabla \cdot \nabla () = \nabla^2 () = \Delta ()$$

In different coordinate systems:

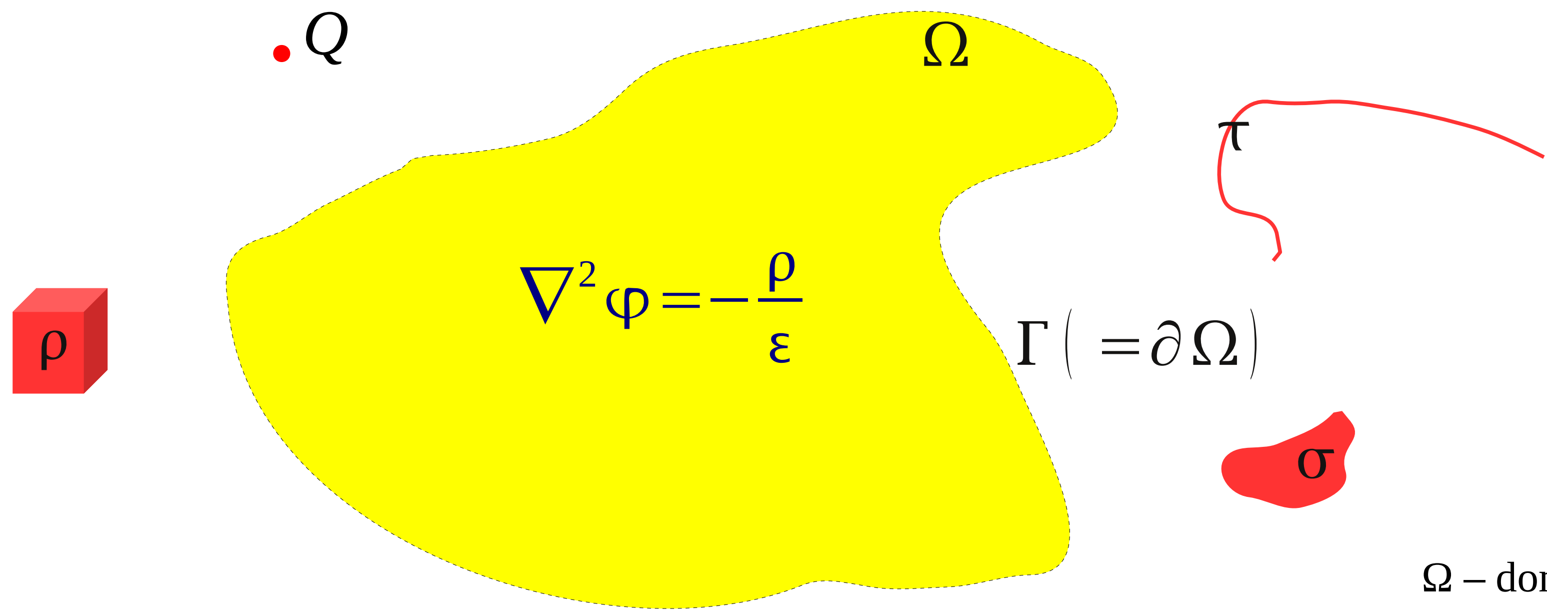
Cartesian: $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

Cylindrical: $\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$

Spherical; $\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial f}{\partial \phi}\right) + \frac{1}{r^2 \sin^2\phi}\frac{\partial^2 f}{\partial \theta^2}$

Laplacian involves partial derivatives ant thus Poisson's or Laplace's equations are called *Partial Derivative Equations (PDE)*

# Solving PDE?




Ω – domain of interest

*Γ* – boundary of *Ω*

*Q,ρ,σ,τ* – external sources

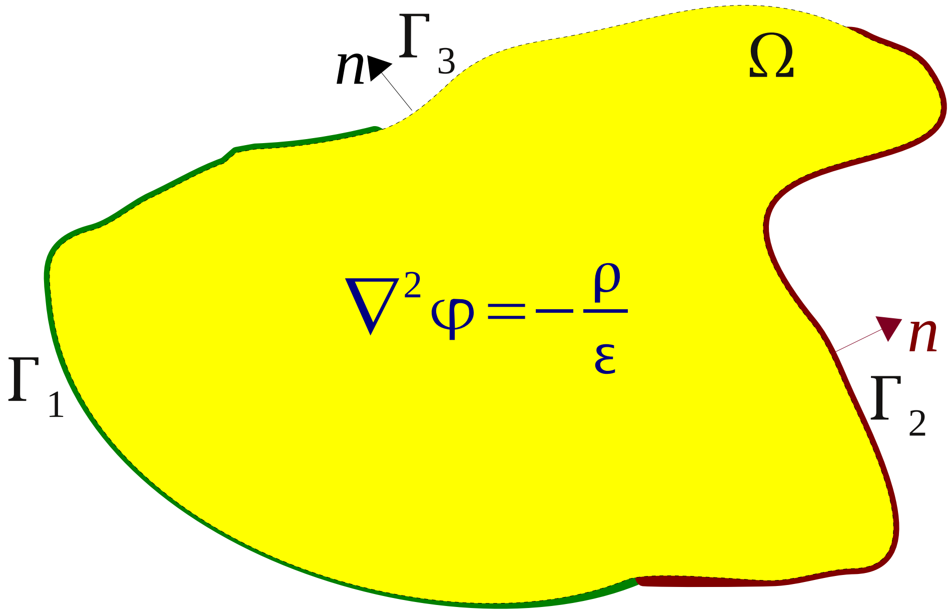
The governing equation (Poisson's equation) describes behavior of field at given point.

It allows us to describe relations of field values at adjacent points, but we are not able to calculate point values not knowing the external sources.

If one wants to restrict analysis to Ω only, then he needs to set on Γ some conditions for φ or it's derivatives.

# Boundary value problems

**Boundary (value) problem = governing equation + boundary conditions**




$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$$

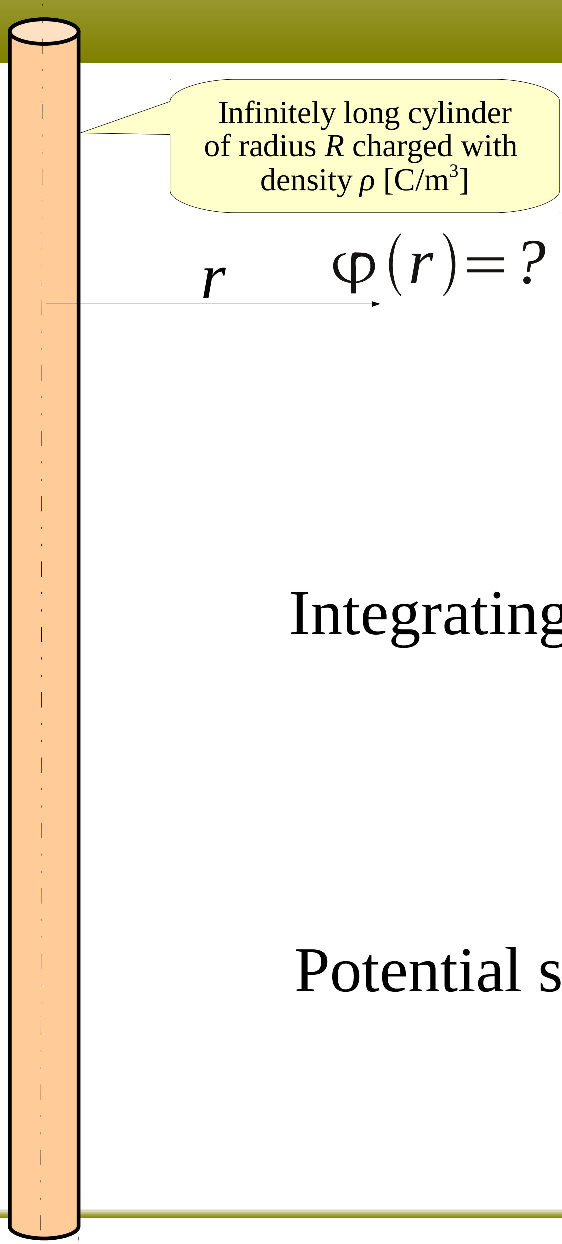
BC types (selected):

1) Dirichlet (1st kind) : $\varphi = u \quad on \quad \Gamma_1$

2) Neumann (2nd kind): $\frac{\partial \varphi}{\partial n} = q \quad on \quad \Gamma_2$

3) Robin (3rd kind, impedance BC): $a\varphi + b\frac{\partial \varphi}{\partial n} = v \quad on \quad \Gamma_3$

# Example: field of charged cylinder




$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{d\varphi_1(r)}{dr}\right)=-\frac{\rho}{\varepsilon_0} \qquad r<R$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{d\varphi_2(r)}{dr}\right)=0 \qquad r\geq R$$

Integrating twice we get

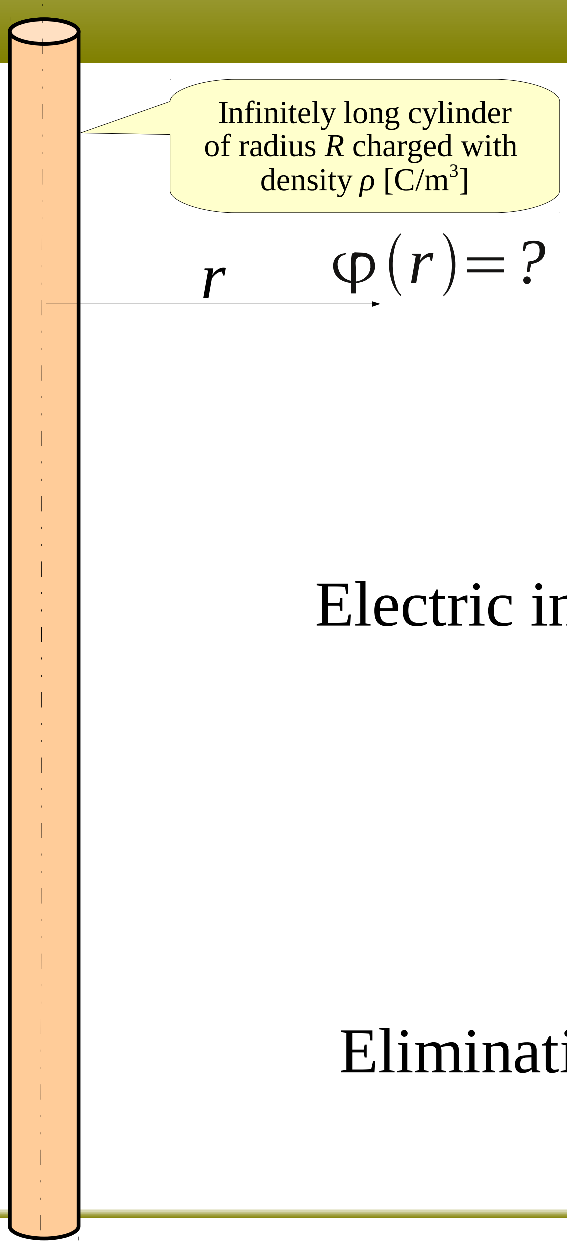
$$\varphi_1(r)=-\frac{\rho}{4\varepsilon_0}r^2+A_1\ln(r)+B_1 \qquad r<R$$

$$\varphi_2(r)=A_2\ln(r)+B_2 \qquad r\geq R$$

Potential should be finite at the cylinder axis ($r=0$):

$$A_1=0, \quad \varphi_1(r)=-\frac{\rho}{4\varepsilon_0}r^2+B_1 \qquad r<R$$

# Example: field of charged cylinder
(continued)




Continuous potential would be more convenient:

$$\varphi_1(r)=\varphi_2(r) \qquad r=R$$

$$-\frac{\rho}{4\varepsilon_0}R^2+B_1=A_2\ln(R)+B_2$$

$$B_2=B_1-\frac{\rho}{4\varepsilon_0}R^2-A_2\ln(R)$$

Electric induction must be continuous:

$$\varepsilon_0\frac{d\varphi_1(r)}{dr}=\varepsilon_0\frac{d\varphi_2(r)}{dr} \qquad r=R$$

$$-\frac{\rho}{2}R=\varepsilon_0\frac{A_2}{R} \quad\rightarrow\quad A_2=-\frac{\rho R^2}{2\varepsilon_0}$$

Eliminating $B_2$:

$$\varphi_2(r)=-\frac{\rho R^2}{2\varepsilon_0}\ln\left(\frac{R}{r}\right)+B_1-\frac{\rho R^2}{4\varepsilon_0} \qquad r\geq R$$



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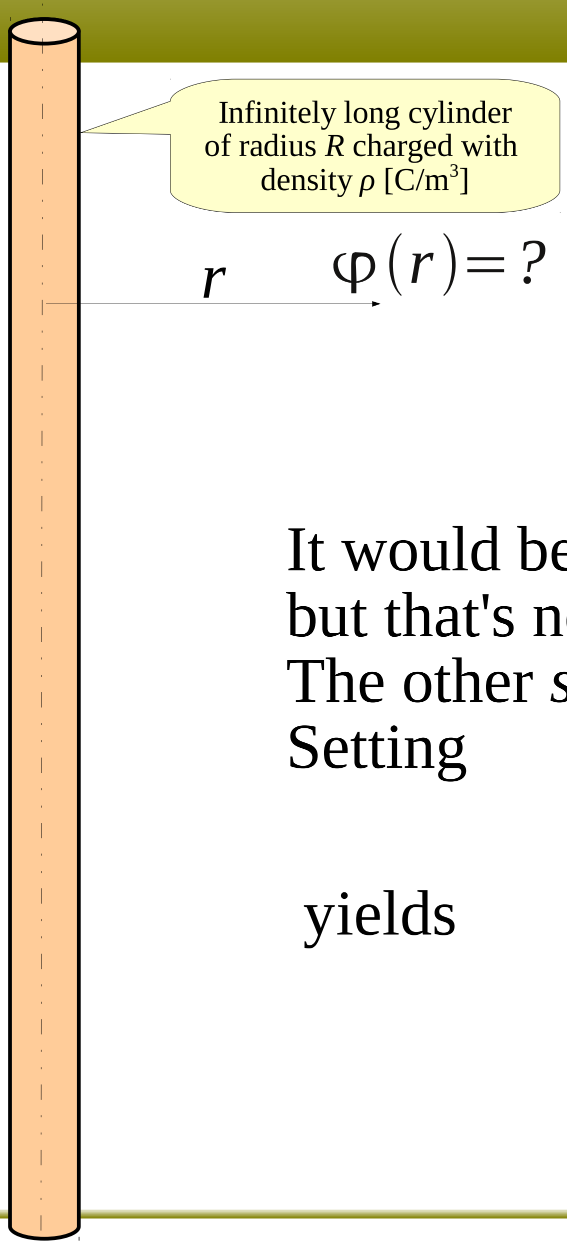
# Example: field of charged cylinder
(continued)




Finalization – choosing the reference potential:

$$\varphi_1(r)=-\frac{\rho}{4\varepsilon_0}r^2+B_1 \qquad r<R$$

$$\varphi_2(r)=-\frac{\rho R^2}{2\varepsilon_0}\ln\left(\frac{R}{r}\right)+B_1-\frac{\rho R^2}{4\varepsilon_0} \qquad r\geq R$$

It would be convenient to postulate *φ(∞)=0*,
but that's not possible because ln(∞) = ∞.
The other *special place* is the outer surface of cylinder *r*= *R*
Setting

$$\varphi_1(R)=\varphi_2(R)=0 \quad \rightarrow \quad B_1=\frac{\rho R^2}{4\varepsilon}$$

yields

$$\varphi_1(r)=-\frac{\rho R^2}{4\varepsilon_0}\left(1-\frac{r^2}{R^2}\right) \qquad r<R$$

$$\varphi_2(r)=-\frac{\rho R^2}{2\varepsilon_0}\ln\left(\frac{R}{r}\right) \qquad r\geq R$$



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